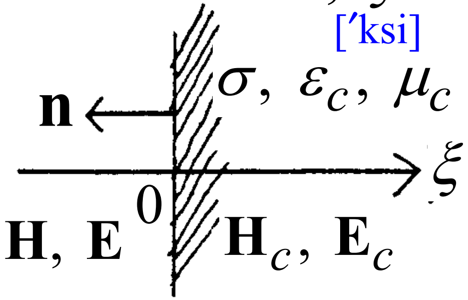
Chapter 8: Waveguides, Resonant Cavities, and Optical Fibers

8.1 Fields at the Surface of and Within a Good Conductor*

Notations: $\mathbf{H},\ \mathbf{E}$: fields outside the conductor; $\mathbf{H}_C,\ \mathbf{E}_C$: fields inside the conductor; $\mathbf{n}$: a unit vector $\perp$ to conductor surface; ξ : a normal coordinate into the conductor. [ˈksi]



Assume: (i) fields $\sim e^{-i\omega t}$ Why?

(ii) good but not perfect conductor, i.e.,

$\sigma \neq \infty$, but $\dfrac{\sigma}{\omega\varepsilon_b} \gg 1$ [See Ch. 7 of lecture notes, Eq. (24)].

(iii) $\mathbf{H}_{||}(\xi = 0)$ is known. Why?

Find: $\mathbf{E}_c(\xi),\ \mathbf{H}_c(\xi),$ and power loss, etc. in terms of $\mathbf{H}_{||}(\xi = 0)$

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*The main results in Sec. 8.1 [(8.9), (8.10), (8.12), (8.14), and (8.15)] have been derived with a much simpler method in Ch. 7 of lecture notes. [See contents following Eq. (26)]. So, we will not cover this section in classes.

Calculation of $\mathbf{E}_C(\xi)$, $\mathbf{H}_C(\xi)$: In the conductor, we have

good conductor assumption

$$\begin{cases} \nabla\times\mathbf{E}_c = -\frac{\partial}{\partial t}\mathbf{B}_c = i\omega\mu_c\mathbf{H}_c & (1) \\ \nabla\times\mathbf{H}_c = \mathbf{J}+\frac{\partial}{\partial t}\mathbf{D}_c = \sigma\mathbf{E}_c - i\omega\varepsilon_b\mathbf{E}_c \simeq \sigma\mathbf{E}_c & (2) \end{cases}$$

$$\nabla \simeq -\mathbf{n}\frac{\partial}{\partial\xi} \quad (3)$$

[In a good conductor, fields vary rapidly along the normal to the surface, see Ch. 7 of lecture notes.]

$$(1), (2), (3) \Rightarrow \begin{cases} \mathbf{E}_c \approx -\frac{1}{\sigma}\mathbf{n}\times\frac{\partial}{\partial\xi}\mathbf{H}_c & (4) \\ \mathbf{H}_c \approx \frac{i}{\mu_C\omega}\mathbf{n}\times\frac{\partial}{\partial\xi}\mathbf{E}_c & (5) \end{cases}$$

$$\mathbf{H}_c \approx \left(\frac{i}{\mu_C\omega}\mathbf{n}\times\frac{\partial}{\partial\xi}\left(-\frac{1}{\sigma}\mathbf{n}\times\frac{\partial}{\partial\xi}\mathbf{H}_c\right)\right)$$

$$= \frac{i}{\mu_C\omega\sigma}(\mathbf{n}\cdot\mathbf{n})\frac{\partial^2}{\partial\xi^2}\mathbf{H}_c - \frac{i}{\mu_C\omega\sigma}\frac{\partial^2}{\partial\xi^2}(\mathbf{n}\cdot\mathbf{H}_c)\mathbf{n}$$

Since $\mathbf{n}\cdot\mathbf{H}_c \approx 0$ and $\mathbf{n}\cdot\mathbf{n} = 1$,

$$\Rightarrow \mathbf{n}\times\mathbf{H}_c = \frac{i}{\mu_C\omega\sigma}\frac{\partial^2}{\partial\xi^2}(\mathbf{n}\times\mathbf{H}_c)$$

$$\delta \equiv \sqrt{\frac{2}{\mu_c\omega\sigma}}$$

skin depth

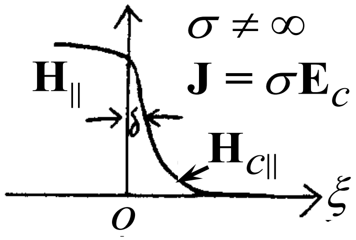
KK[dεpθ]

Substituting (4) into (5): $\frac{\partial^2}{\partial\xi^2}(\mathbf{n}\times\mathbf{H}_c) + \frac{2i}{\delta^2}(\mathbf{n}\times\mathbf{H}_c) \approx 0$ (8.7)

$$\Rightarrow \mathbf{n}\times\mathbf{H}_c(\xi) \approx \mathbf{n}\times\mathbf{H}_c(0)e^{-\frac{\xi}{\delta}}e^{\frac{i\xi}{\delta}}$$

b.c. at $\xi = 0$: $\mathbf{H}_{||}(0) = \mathbf{H}_{c||}(0)$

$$\Rightarrow \mathbf{H}_{c||}(\xi) \approx \mathbf{H}_{c||}(0)e^{-\frac{\xi}{\delta}}e^{\frac{i\xi}{\delta}} = \mathbf{H}_{||}(0)e^{-\frac{\xi}{\delta}}e^{\frac{i\xi}{\delta}} \quad (6)$$



$\sigma \neq \infty$

$\mathbf{J} = \sigma\mathbf{E}_c$

$$\mathbf{n}\cdot(5) \Rightarrow \mathbf{n}\cdot\mathbf{H}_c(\xi) \approx 0 \Rightarrow \mathbf{H}_{c||}(\xi) \approx \mathbf{H}_c(\xi) \quad (7)$$

Substituting (7) into (6) $\Rightarrow \mathbf{H}_c(\xi) \approx \mathbf{H}_{||}(0)e^{-\frac{\xi}{\delta}}e^{\frac{i\xi}{\delta}}$ (8.9)

Substituting $\mathbf{H}_c(\xi) \approx \mathbf{H}_{\|}(0) e^{-\frac{\xi}{\delta}} e^{\frac{i\xi}{\delta}}$ into $\mathbf{E}_c(\xi) \approx -\frac{1}{\sigma} \mathbf{n} \times \frac{\partial}{\partial \xi} \mathbf{H}_c(\xi)$

$$\Rightarrow \mathbf{E}_c(\xi) \approx \sqrt{\frac{\mu_C \omega}{2\sigma}} (1-i) \left[\mathbf{n} \times \mathbf{H}_{\|}(\xi = 0) \right] e^{-\frac{\xi}{\delta}} e^{\frac{i\xi}{\delta}} \qquad (8.10)$$

$$\mathbf{E}_{\|}(\xi = 0) = \mathbf{E}_{c\|}(\xi = 0) \approx \mathbf{E}_c(\xi = 0) \approx \sqrt{\frac{\mu_C \omega}{2\sigma}} (1-i) \left[\mathbf{n} \times \mathbf{H}_{\|}(\xi = 0) \right] \qquad (8.11)$$

b.c. at $\xi = 0$ ↑

$\mathbf{n} \cdot (4) \Rightarrow \mathbf{n} \cdot \mathbf{E}_C \simeq 0 \Rightarrow \mathbf{E}_{C\|} \simeq \mathbf{E}_C$ ↑

$\mathbf{E}_{\perp}(\xi = 0) = ?$

Power Loss Per Unit Area :

$$\frac{dP_{loss}}{da} = \text{time averaged power into conductor per unit area}$$

$$= -\frac{1}{2} \mathrm{Re} \left[\mathbf{n} \cdot \mathbf{E}(\xi = 0) \times \mathbf{H}^*(\xi = 0) \right]$$

$$= -\frac{1}{2} \mathrm{Re} \left[\mathbf{n} \cdot \mathbf{E}_{\|}(\xi = 0) \times \mathbf{H}_{\|}^*(\xi = 0) \right]$$

(8.11) →

$$= \frac{1}{4} \mu_C \omega \delta \left| \mathbf{H}_{\|}(\xi = 0) \right|^2 = \frac{1}{2\sigma\delta} \left| \mathbf{H}_{\|}(\xi = 0) \right|^2 \qquad (8.12)$$

$$\propto \mu_C^{\frac{1}{2}} \omega^{\frac{1}{2}} \sigma^{-\frac{1}{2}} \left| \mathbf{H}_{\|}(\xi = 0) \right|^2$$

Alternative method to derive (8.12):

$$(8.10) \Rightarrow \mathbf{J}(\xi) = \sigma \mathbf{E}_c(\xi) \approx \tfrac{1}{\delta}(1-i)\left[\mathbf{n}\times\mathbf{H}_{\|}(\xi=0)\right]e^{-\frac{\xi(1-i)}{\delta}} \qquad (8.13)$$

$$\begin{bmatrix}\text{time averaged power}\\ \text{loss in conductor per}\\ \text{unit volume}\end{bmatrix} = \tfrac{1}{2}\mathrm{Re}\left[\mathbf{J}(\xi)\cdot\mathbf{E}_c^*(\xi)\right] = \tfrac{1}{2\sigma}\left|\mathbf{J}(\xi)\right|^2$$

(8.13)

$$\frac{dP_{loss}}{da} = \frac{1}{2\sigma}\int_0^\infty d\xi \left|\mathbf{J}(\xi)\right|^2 \overset{(8.13)}{=} \frac{1}{\sigma\delta^2}\left|\mathbf{H}_{\|}(\xi=0)\right|^2 \int_0^\infty e^{-\frac{2\xi}{\delta}}d\xi$$

$$= \frac{1}{2\sigma\delta}\left|\mathbf{H}_{\|}(\xi=0)\right|^2, \quad \text{same as (8.12)}$$

Effective surface current $\mathbf{K}_{eff}$:

$$\mathbf{K}_{eff} = \int_0^\infty \mathbf{J}(\xi)d\xi = \tfrac{1}{\delta}(1-i)\left[\mathbf{n}\times\mathbf{H}_{\|}(\xi=0)\right]\int_0^\infty e^{-\frac{\xi(1-i)}{\delta}}d\xi$$

$$= \mathbf{n}\times\mathbf{H}_{\|}(\xi=0) \qquad (8.14)$$

(8.13)

$$(8.12)\ \&\ (8.14) \Rightarrow \frac{dP_{loss}}{da} = \frac{1}{2\sigma\delta}\left|\mathbf{K}_{eff}\right|^2 \qquad (8.15)$$

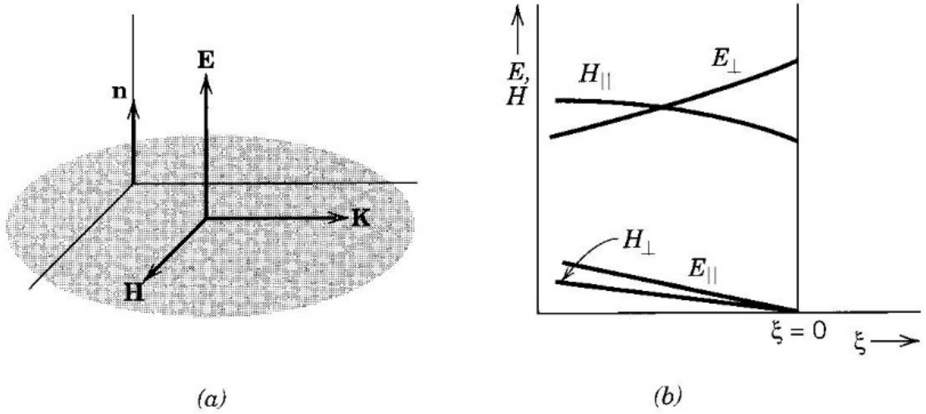


Figure 8.1 Fields near the surface of a perfect conductor.

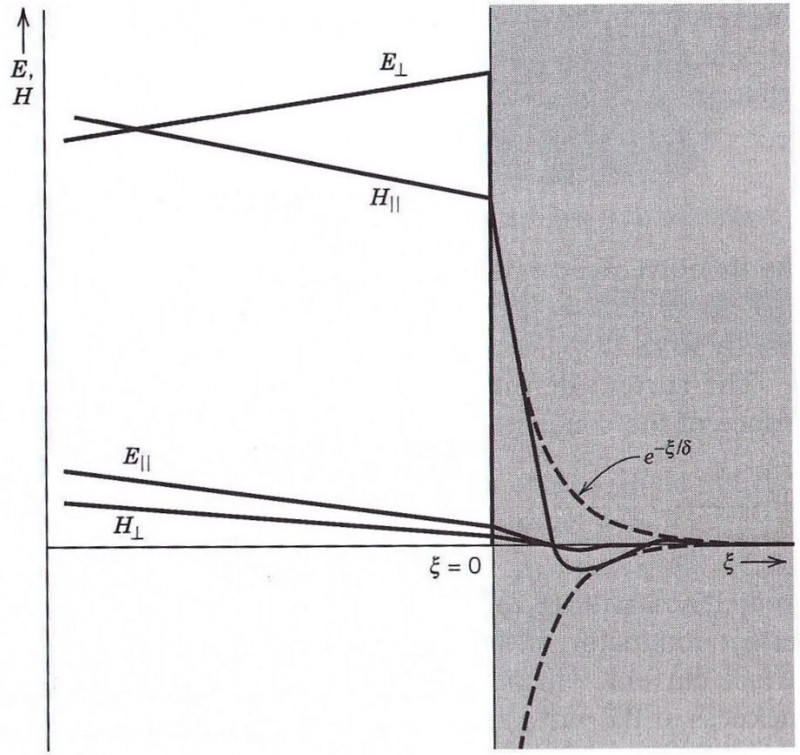
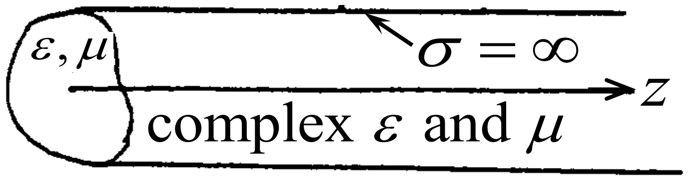


Figure 8.2 Fields near the surface of a good, but not perfect, conductor. For $\xi > 0$, the dashed curves show the envelope of the damped oscillations of $\mathbf{H}_c$ (8.9).

8.2-8.4 Modes in a Waveguide

Consider a hollow conductor of infinite length and uniform cross section of arbitrary shape (see figure). We assume that the filling medium is uniform, linear, and isotropic ($\mathbf{B}=\mu\mathbf{H};\ \mathbf{D}=\varepsilon\mathbf{E},$ where ε and μ are in general complex numbers). This is a structure commonly used to guide EM waves as well as a rare case where exact solutions are possible (for some simple cross sections.) Maxwell equations can be written

$$\begin{cases}\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{B} & (8)\\ \nabla\times\mathbf{B}=\mu\varepsilon\frac{\partial}{\partial t}\mathbf{E} & (9)\\ \nabla\cdot\mathbf{E}=0 & (10)\\ \nabla\cdot\mathbf{B}=0 & (11)\end{cases}$$



$$\nabla\times(8)\Rightarrow\nabla\times\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\nabla\times\mathbf{B}\Rightarrow\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}=-\frac{\partial}{\partial t}\left(\mu\varepsilon\frac{\partial}{\partial t}\mathbf{E}\right)$$

$$\Rightarrow\nabla^2\mathbf{E}-\mu\varepsilon\frac{\partial^2}{\partial t^2}\mathbf{E}=0 \qquad (12)$$

Similarly, $\nabla\times(9)\Rightarrow\nabla^2\mathbf{B}-\mu\varepsilon\frac{\partial^2}{\partial t^2}\mathbf{B}=0$ (13)

Let $\begin{cases} \mathbf{E}(\mathbf{x},t) = \mathbf{E}(\mathbf{x}_t)e^{\pm ik_z z - i\omega t} \\ \mathbf{B}(\mathbf{x},t) = \mathbf{B}(\mathbf{x}_t)e^{\pm ik_z z - i\omega t} \end{cases}$,

> $\mathbf{x}_t$: coordinates transverse to z, e.g., (x, y) or (r, θ)
> k_z here $\leftrightarrow$ k in Jackson

where, in general, ω and k_z are complex constants. To be specific, we assume that the real parts of ω and k_z are both positive. Then, $e^{ik_z z - i\omega t}$ and $e^{-ik_z z - i\omega t}$ have forward and backward phase velocities, respectively. As will be seen in (31), $e^{ik_z z - i\omega t}$ and $e^{-ik_z z - i\omega t}$ also have forward and backward group velocities, respectively. Hence, we call $e^{ik_z z - i\omega t}$ a forward wave and $e^{-ik_z z - i\omega t}$ a backward wave.

With the assumed z and t dependences, we have

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} \to -\omega^2 \\ \dfrac{\partial^2}{\partial z^2} \to -k_z^2 \\ \nabla^2 = \nabla_t^2 + \dfrac{\partial^2}{\partial z^2} = \nabla_t^2 - k_z^2 \end{cases}$$

$$\nabla_t^2 = \begin{cases} \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}, & \text{Cartesian} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2}, & \text{cylindrical} \end{cases}$$

Thus,

$$\begin{cases} \nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2}{\partial t^2} \mathbf{E} = 0 \\ \nabla^2 \mathbf{B} - \mu\varepsilon \frac{\partial^2}{\partial t^2} \mathbf{B} = 0 \end{cases} \Rightarrow \left(\nabla_t^2 + \mu\varepsilon\omega^2 - k_z^2 \right) \begin{Bmatrix} \mathbf{E}(\mathbf{x}_t) \\ \mathbf{B}(\mathbf{x}_t) \end{Bmatrix} = 0 \qquad (8.19)$$

$$\Rightarrow \left(\nabla_t^2 + \mu\varepsilon\omega^2 - k_z^2 \right) \begin{Bmatrix} E_z(\mathbf{x}_t) \\ B_z(\mathbf{x}_t) \end{Bmatrix} = 0 \qquad (14)$$

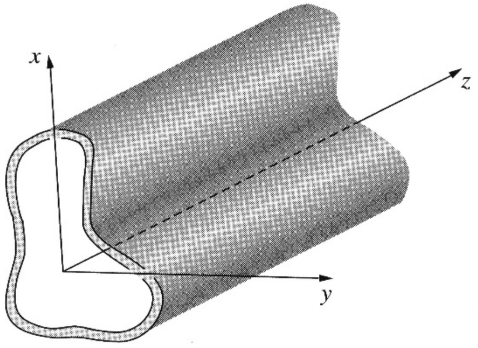
It is in general not possible to obtain from (8.19). So our strategy here is to solve (14) for $E_z(\mathbf{x}_t)$ and $B_z(\mathbf{x}_t)$, and then express the other components of the fields [$\mathbf{E}_t(\mathbf{x}_t)$ and $\mathbf{B}_t(\mathbf{x}_t)$] in terms of $E_z(\mathbf{x}_t)$ and $B_z(\mathbf{x}_t)$ through Eqs. (17) and (18).

Exercise: Writing $\mathbf{E}(\mathbf{x}_t) = E_r\mathbf{e}_r + E_\theta\mathbf{e}_\theta + E_z\mathbf{e}_z$ and using the cylindrical coordinate system, derive the equations for E_r and E_θ from (8.19).
(hint: $\frac{\partial}{\partial\theta}\mathbf{e}_r = \mathbf{e}_\theta, \ \ \frac{\partial}{\partial\theta}\mathbf{e}_\theta = -\mathbf{e}_r$)

Griffiths

9.5 Guided Waves
9.5.1 Wave Guides

Can the electromagnetic waves propagate in a hollow metal pipe?
Yes, wave guide.



Waveguides generally made of good conductor, so that $\mathbf{E}=0$ and $\mathbf{B}=0$ inside the material.

The boundary conditions at the inner wall are: $\mathbf{E}^{//}=0$ and $B^{\perp}=0$ …

The generic form of the monochromatic waves:

$$\begin{cases} \tilde{\mathbf{E}}(x,y,z,t)=\tilde{\mathbf{E}}_0(x,y)e^{i(\tilde{k}z-\omega t)}=(\tilde{E}_x\hat{\mathbf{x}}+\tilde{E}_y\hat{\mathbf{y}}+\tilde{E}_z\hat{\mathbf{z}})e^{i(\tilde{k}z-\omega t)} \\ \tilde{\mathbf{B}}(x,y,z,t)=\tilde{\mathbf{B}}_0(x,y)e^{i(\tilde{k}z-\omega t)}=(\tilde{B}_x\hat{\mathbf{x}}+\tilde{B}_y\hat{\mathbf{y}}+\tilde{B}_z\hat{\mathbf{z}})e^{i(\tilde{k}z-\omega t)} \end{cases}$$

Griffiths

General Properties of Wave Guides

In the interior of the wave guide, the waves satisfy Maxwell's equations:

$$\nabla \cdot \mathbf{E} = 0 \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

Why $\rho_f = 0$ and $J_f = 0$?

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

where $c = \frac{1}{\sqrt{\varepsilon\mu}}$

We obtain

$$\text{(i)}\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z \qquad \text{(iv)}\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2} E_z$$

$$\text{(ii)}\ \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = i\omega B_x \qquad \text{(v)}\ \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = -\frac{i\omega}{c^2} E_x$$

$$\text{(iii)}\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = i\omega B_y \qquad \text{(vi)}\ \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2} E_y$$

Griffiths

TE, TM, and TEM Waves

Determining the longitudinal components E_z and B_z, we could quickly calculate all the others.

$$E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

$$E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right)$$

$$B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

$$B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

Try to derive these relations by yourself.

We obtain

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right]E_z = 0$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\omega^2}{c^2} - k^2\right]B_z = 0$$

If $E_z = 0 \Rightarrow$ TE (transverse electric) waves;

If $B_z = 0 \Rightarrow$ TM (transverse magnetic) waves;

If $E_z = 0$ and $B_z = 0 \Rightarrow$ TEM waves.

General Approach

Let $\begin{cases} \mathbf{E} = \mathbf{E}_t + E_z \mathbf{e}_z \\ \mathbf{B} = \mathbf{B}_t + B_z \mathbf{e}_z \\ \nabla = \nabla_t + \mathbf{e}_z \frac{\partial}{\partial z} = \nabla_t \pm i k_z \mathbf{e}_z \end{cases}$

$$\nabla_t = \begin{cases} \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y}, & \text{Cartesian} \\ \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta}, & \text{cylindrical} \end{cases}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B} \quad \Rightarrow (\nabla_t \pm i k_z \mathbf{e}_z) \times (\mathbf{E}_t + E_z \mathbf{e}_z) = i\omega (\mathbf{B}_t + B_z \mathbf{e}_z) \qquad (15)$$

$$\nabla \times \mathbf{B} = \mu\varepsilon \frac{\partial}{\partial t} \mathbf{E} \Rightarrow (\nabla_t \pm i k_z \mathbf{e}_z) \times (\mathbf{B}_t + B_z \mathbf{e}_z) = -i\mu\varepsilon\omega (\mathbf{E}_t + E_z \mathbf{e}_z) \quad (16)$$

Using the relations: $\begin{Bmatrix} (\nabla_t \times \mathbf{E}_t) \parallel \mathbf{e}_z \\ (\nabla_t \times E_z \mathbf{e}_z) \perp \mathbf{e}_z \end{Bmatrix}$, we obtain from the transverse components of (15) and (16):

$$\nabla_t \times E_z \mathbf{e}_z \pm i k_z \mathbf{e}_z \times \mathbf{E}_t = i\omega \mathbf{B}_t \qquad (17)$$

$$\nabla_t \times B_z \mathbf{e}_z \pm i k_z \mathbf{e}_z \times \mathbf{B}_t = -i\mu\varepsilon\omega \mathbf{E}_t \qquad (18)$$

In (15)-(18), the $\begin{Bmatrix} \text{upper} \\ \text{lower} \end{Bmatrix}$ sign applies to the $\begin{Bmatrix} \text{forward} \\ \text{backward} \end{Bmatrix}$ wave.

Rewrite (17) and (18)

$$\nabla_t \times E_z \mathbf{e}_z \pm i k_z \mathbf{e}_z \times \mathbf{E}_t = i\omega \mathbf{B}_t \qquad (17)$$

$$\nabla_t \times B_z \mathbf{e}_z \pm i k_z \mathbf{e}_z \times \mathbf{B}_t = -i\mu\varepsilon\omega \mathbf{E}_t \qquad (18)$$

Since E_z and B_z have already been solved from (14), (17) and (18) are algebraic (rather than differential) equations. We now manipulate (17) and (18) to eliminate $\mathbf{B}_t$ and thus express $\mathbf{E}_t$ in terms of E_z and B_z.

$$\mathbf{e}_z \times (17) \Rightarrow \ \mathbf{e}_z \times \underbrace{(\nabla_t \times E_z \mathbf{e}_z)}_{\nabla_t E_z \times \mathbf{e}_z + E_z \underbrace{\nabla_t \times \mathbf{e}_z}_{0}} \pm i k_z \overbrace{\mathbf{e}_z \times (\mathbf{e}_z \times \mathbf{E}_t)}^{-\mathbf{E}_t} = i\omega \mathbf{e}_z \times \mathbf{B}_t$$

$\nabla \times \psi \mathbf{a} = \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a}$
If ψ, $\mathbf{a}$ are both independent of z, then
$\nabla_t \times \psi \mathbf{a} = \nabla_t \psi \times \mathbf{a} + \psi \nabla_t \times \mathbf{a}$

$$\Rightarrow \ i\omega \mathbf{e}_z \times \mathbf{B}_t = \nabla_t E_z \mp i k_z \mathbf{E}_t \qquad (19)$$

Substituting (19) into (18)

$$\underbrace{\nabla_t \times B_z \mathbf{e}_z}_{\nabla_t B_z \times \mathbf{e}_z} \pm ik_z \frac{1}{i\omega}\left(\nabla_t E_z \mp ik_z \mathbf{E}_t\right) = -i\mu\varepsilon\omega\mathbf{E}_t \qquad (20)$$

Multiply (20) by $i\omega$: $i\omega\nabla_t B_z \times \mathbf{e}_z \pm ik_z\nabla_t E_z + k_z^2\mathbf{E}_t = \mu\varepsilon\omega^2\mathbf{E}_t$

$$\Rightarrow \left(\mu\varepsilon\omega^2 - k_z^2\right)\mathbf{E}_t = i\left(\omega\nabla_t B_z \times \mathbf{e}_z \pm k_z\nabla_t E_z\right)$$

$$\Rightarrow \mathbf{E}_t = \frac{i}{\mu\varepsilon\omega^2 - k_z^2}\left[\pm k_z\nabla_t E_z - \omega\mathbf{e}_z \times \nabla_t B_z\right] \qquad (8.26a)$$

Similarly,

$$\mathbf{B}_t = \frac{i}{\mu\varepsilon\omega^2 - k_z^2}\left[\pm k_z\nabla_t B_z + \mu\varepsilon\omega\mathbf{e}_z \times \nabla_t E_z\right] \qquad (8.26b)$$

Thus, once E_z and B_z have been solved from (14), the solutions for $\mathbf{E}_t$ and $\mathbf{B}_t$ are given by (8.26a) and (8.26b).

Discussion:

(i) $\mathbf{E}_t$, $\mathbf{B}_t$, E_z, B_z in (8.26a) and (8.26b) are functions of $\mathbf{x}_t$ only.

(ii) ε and μ can be complex. Im(ε) or Im(μ) implies dissipation.

(iii) By letting $B_z = 0$, we may obtain a set of solutions for E_z, $\mathbf{E}_t$, and $\mathbf{B}_t$ from (14), (8.26a), and (8.26b), respectively. It can be shown that if the boundary condition on E_z is satisfied, then boundary conditions on $\mathbf{E}_t$ and $\mathbf{B}_t$ are also satisfied. Hence, this gives a set of valid solutions called the TM (transverse magnetic) modes. Similarly, by letting $E_z = 0$, we may obtain a set of valid solutions called the TE (transverse electric) modes.

(iv) E_z is the generating function for the TM mode and B_z is the generating function for the TE mode. The generating function is denoted by Ψ in Jackson.

8.2-8.4 Modes in Waveguides (*continued*)

TM Mode of a Waveguide ($B_z = 0$): (see pp. 359-360)

$$\begin{cases} (\nabla_t^2 + \gamma^2) E_z = 0 \text{ with boundary condition } E_z\big|_s = 0 & (21) \\ \mathbf{E}_t = \pm \dfrac{ik_z}{\gamma^2} \nabla_t E_z & (21a) \\ \mathbf{H}_t = \pm \dfrac{\varepsilon\omega}{k_z} \mathbf{e}_z \times \mathbf{E}_t = \pm \dfrac{1}{Z_e} \mathbf{e}_z \times \mathbf{E}_t & (21b) \\ \gamma^2 = \mu\varepsilon\omega^2 - k_z^2 & (21c) \end{cases}$$

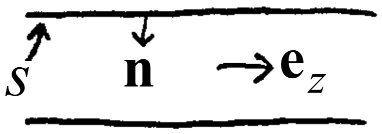
Assume perfectly conducting wall.

$Z_e \equiv k_z / \varepsilon\omega$, wave impedance of TM modes

TE Mode of a Waveguide ($E_z = 0$): (see pp. 359-360)

$$\begin{cases} (\nabla_t^2 + \gamma^2) H_z = 0 \text{ with boundary condition } \dfrac{\partial}{\partial n} H_z\big|_s = 0 & (22) \\ \mathbf{H}_t = \pm \dfrac{ik_z}{\gamma^2} \nabla_t H_z & (22a) \\ \mathbf{E}_t = \mp \dfrac{\mu\omega}{k_z} \mathbf{e}_z \times \mathbf{H}_t = \mp Z_h \mathbf{e}_z \times \mathbf{H}_t & (22b) \\ \gamma^2 = \mu\varepsilon\omega^2 - k_z^2 & (22c) \end{cases}$$

$Z_h \equiv \mu\omega / k_z$, wave impedance of TE modes



b.c. $\mathbf{n} \cdot \mathbf{H}\big|_s = 0$

$\mathbf{n} \perp \mathbf{e}_z \Rightarrow \mathbf{n} \cdot \mathbf{H}_t\big|_s = 0$

(22a) $\Rightarrow \mathbf{n} \cdot \nabla_t H_z\big|_s = 0$

$\Rightarrow \frac{\partial}{\partial n} H_z\big|_s = 0$

Discussion:

(i) Either (21) or (22) constitutes an eigenvalue problem (see lecture notes, Ch. 3, Appendix A). The eigenvalue γ^2 will be an infinite set of discrete values fixed by the boundary condition, each representing an eigenmode of the waveguide (An example will be provided below.)

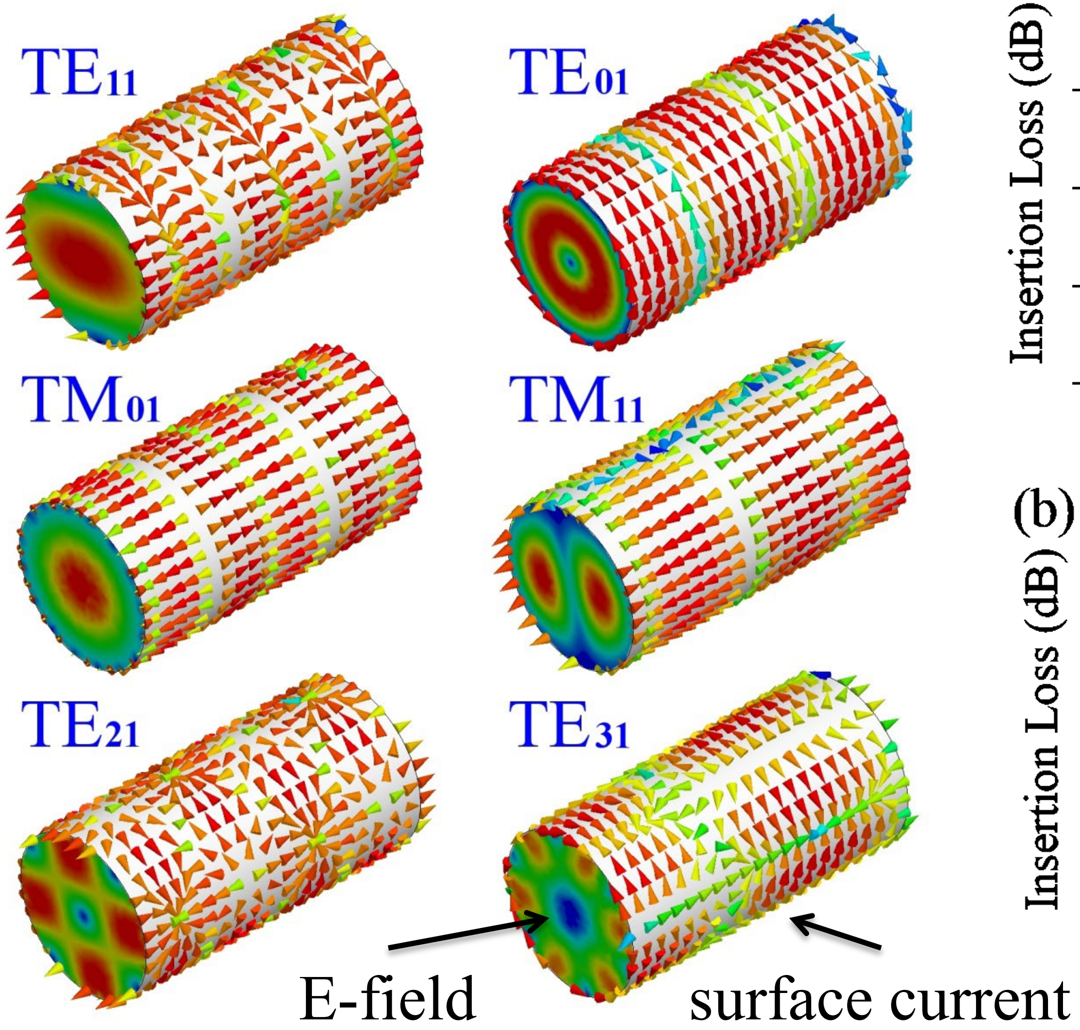
(ii) (21b) and (22b) show that $\mathbf{E}_t$ is perpendicular to $\mathbf{B}_t$ (also true in a cavity).

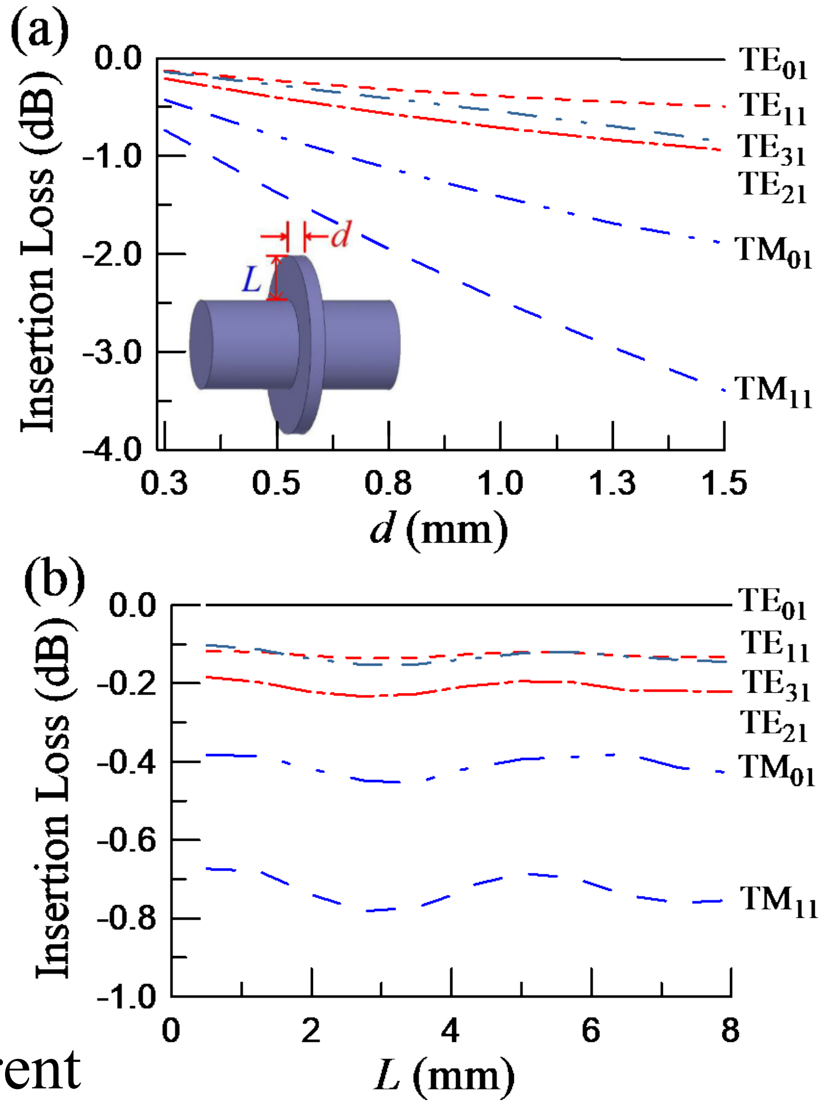
(iii) (21b) and (22b) show that $\mathbf{E}_t$ and $\mathbf{B}_t$ are *in phase* if μ, ε, ω, k_z are all real (not true in a cavity).

(iv) (21c) [or (22c)] is the dispersion relation, which relates ω and k_z for a given mode.

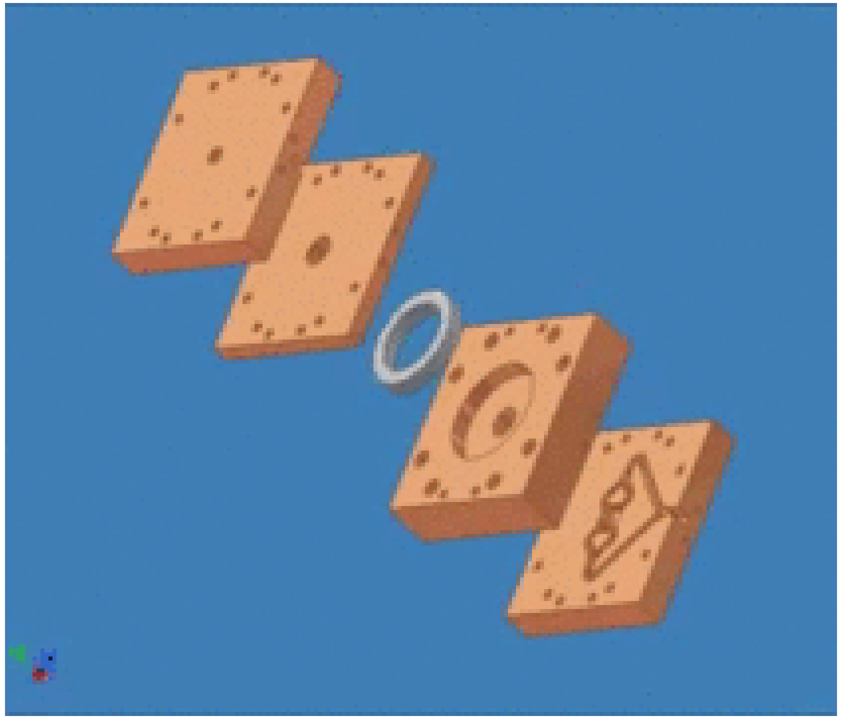
(v) The wave impedance, Z_e or Z_h, gives the ratio of E_t to H_t in the waveguide.

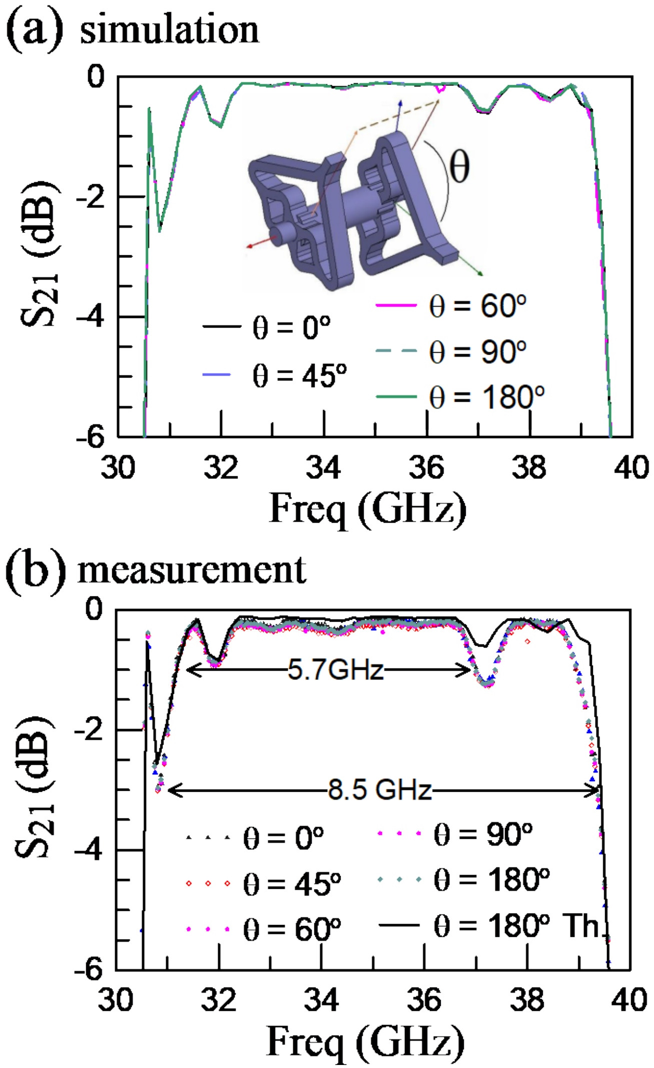
Field Patterns of Circular Waveguide Modes





Characterization of Circularly Symmetric TE_{01} Mode





Rev. Sci. Instrum. **80**, 034701 (2009).

TEM Mode of Coaxial and Parallel-Wire Transmission Lines

$(E_z = B_z = 0)$: (see Jackson p. 341)

Rewrite
$$\begin{cases} \mathbf{E}_t = \dfrac{i}{\mu\varepsilon\omega^2 - k_z^2}\left[\pm k_z \nabla_t E_z - \omega \mathbf{e}_z \times \nabla_t B_z\right] & \text{(8.26a)} \\ \mathbf{B}_t = \dfrac{i}{\mu\varepsilon\omega^2 - k_z^2}\left[\pm k_z \nabla_t B_z + \mu\varepsilon\omega \mathbf{e}_z \times \nabla_t E_z\right] & \text{(8.26b)} \end{cases}$$

These 2 equations fail for a different class of modes, called the TEM (transverse electromagnetic) mode, for which $E_z = B_z = 0$. However, they give the condition for the existence of this mode:

$$\boxed{\omega^2 = k_z^2 / \mu\varepsilon}. \quad \text{(8.27)}$$

[Equations in rectangular boxes are basic equations for the TEM mode.]

(8.27) is also the dispersion relation in infinite space. This makes the TEM mode very useful because it can propagate at any frequency.

To calulate $\mathbf{E}_t$ and $\mathbf{B}_t$, we need to go back to Maxwell equations.

Let $E_z = B_z = 0$ and $\begin{Bmatrix} \mathbf{E}_t \\ \mathbf{B}_t \end{Bmatrix} = \begin{Bmatrix} \mathbf{E}_{\text{TEM}}(\mathbf{x}_t) \\ \mathbf{B}_{\text{TEM}}(\mathbf{x}_t) \end{Bmatrix} e^{\pm i k_z z - i\omega t}$,

then, because $B_z = 0$, the z-component of $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$ gives

$$\nabla_t \times \mathbf{E}_{\text{TEM}} = 0 \;\Rightarrow\; \boxed{\mathbf{E}_{\text{TEM}} = -\nabla_t \Phi_{\text{TEM}}(\mathbf{x}_t)},$$

$$\boxed{\begin{gathered} \nabla_t \times \mathbf{A}_t(\mathbf{x}_t) = 0 \\ \Updownarrow \\ \mathbf{A}_t(\mathbf{x}_t) = -\nabla_t \Phi(\mathbf{x}_t) \end{gathered}}$$

and, because $E_z = 0$, $\nabla \cdot \mathbf{E} = 0$ gives

$$\nabla_t \cdot \mathbf{E}_{\text{TEM}} = 0 \;\Rightarrow\; \boxed{\nabla_t^2 \Phi_{\text{TEM}}(\mathbf{x}_t) = 0},$$

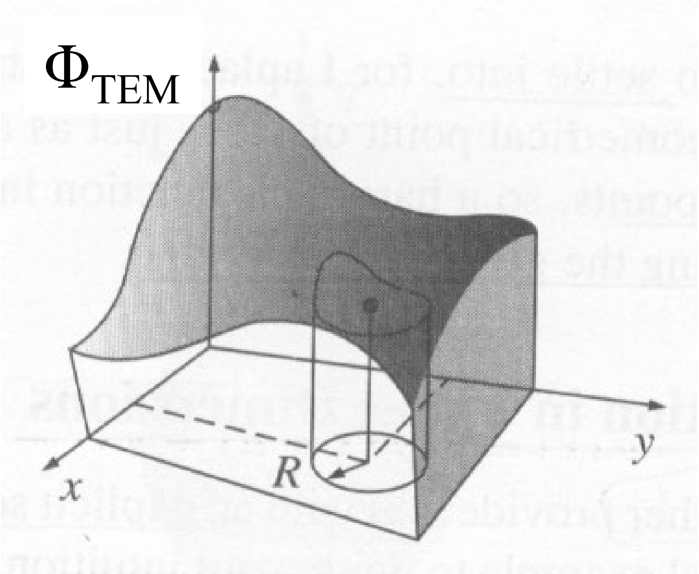
where Φ_{TEM} is the generating function for the TEM modes. Because $\mathbf{E}_{\tan} = 0$ on the surface of a perfect conductor, Φ_{TEM} is subject to the boundary condition $\Phi_{\text{TEM}} = const.$ on the conductor. This gives $\Phi_{\text{TEM}} = const.$ or $\mathbf{E}_{\text{TEM}} = 0$ everywhere, if there is only one conductor. So, TEM modes exist only in 2-conductor configurations, such as coaxial and parallel-wire transmission lines. Finally, $\mathbf{B}_{\text{TEM}}$ is given by the transverse components of $\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$: $\boxed{\mathbf{B}_{\text{TEM}} = \pm \frac{k_z}{\omega} \mathbf{e}_z \times \mathbf{E}_{\text{TEM}}}$.

Why single conductor cannot support TEM waves? (I)

Let's consider the property of 2D Laplace equation.
Suppose Φ_{TEM} depends on two variables.

$$\frac{\partial^2 \Phi_{TEM}}{\partial x^2} + \frac{\partial^2 \Phi_{TEM}}{\partial y^2} = 0 \quad \begin{cases} \text{a partial differential equation (PDE);} \\ \text{not an ordinary differential equation (ODE).} \end{cases}$$

Harmonic functions in two dimensions have the same properties as we noted in one dimension:



Φ_{TEM} has no local maxima or minima. All extrema occur at the boundaries. (The surface may not be an equal potential.)

If Φ_{TEM}=const., $\mathbf{E}_{TEM}$=0 & $\mathbf{B}_{TEM}$=0.

Why single conductor cannot support TEM waves? (II)

David Cheng's explanation. Chap. 10, p.525.

Field and Wave Electromagnetics

1. The magnetic flux lines always close upon themselves. For a TEM wave, the magnetic field line would form closed loops in a transverse plane. (B_z=0)
2. The generalized Ampere's law requires that the line integral of the magnetic field around any closed loop in a transverse plane must equal the sum of the longitudinal conduction and displace conduction current inside the waveguide.
3. There is no longitudinal conduction current inside the waveguide and no longitudinal displace current (E_z=0).
4. There can be no closed loops of magnetic field lines in any transverse plane.

   The TEM wave cannot exist in a single-conductor hollow waveguide of any shape.

In summary, the TEM modes are governed by the following set of equtions:

$$\begin{cases} \nabla_t^2 \Phi_{\text{TEM}}(\mathbf{x}_t) = 0 & (23) \\ \mathbf{E}_{\text{TEM}} = -\nabla_t \Phi_{\text{TEM}}(\mathbf{x}_t) & (23a) \\ \mathbf{B}_{\text{TEM}} = \pm \frac{k_z}{\omega} \mathbf{e}_z \times \mathbf{E}_{\text{TEM}} & (23b) \\ \left(\text{or } \mathbf{H}_{\text{TEM}} = \pm \frac{k_z}{\omega\mu} \mathbf{e}_z \times \mathbf{E}_{\text{TEM}} = \pm \sqrt{\frac{\varepsilon}{\mu}} \mathbf{e}_z \times \mathbf{E}_{\text{TEM}} = \pm \frac{1}{Z} \mathbf{e}_z \times \mathbf{E}_{\text{TEM}}\right) & \\ \omega^2 = \frac{k_z^2}{\mu\varepsilon} & (23c) \end{cases}$$

where $Z \; (= \sqrt{\mu / \varepsilon})$ is the intrinsic impedance of the filling medium defined in Ch. 7 of lecture notes (the last page of Sec. II).

Discussion:

(i) For the TEM modes, we solve a 2-D equation $\nabla_t^2 \Phi_{\text{TEM}}(\mathbf{x}_t) = 0$ for $\Phi_{\text{TEM}}(\mathbf{x}_t)$. But this is not a 2-D problem because Φ_{TEM} is not the full solution. The full solution is $\begin{Bmatrix} \mathbf{E}_t(\mathbf{x},t) \\ \mathbf{B}_t(\mathbf{x},t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{E}_{\text{TEM}}(\mathbf{x}_t) \\ \mathbf{B}_{\text{TEM}}(\mathbf{x}_t) \end{Bmatrix} e^{\pm ik_z z - i\omega t}$ with $\mathbf{E}_{\text{TEM}} = -\nabla_t \Phi_{\text{TEM}}(\mathbf{x}_t)$ and $\mathbf{B}_{\text{TEM}} = \pm \frac{k_z}{\omega} \mathbf{e}_z \times \mathbf{E}_{\text{TEM}}$.

For an actual 2-D electrostatic problem $[\Phi(\mathbf{x}) = \Phi(\mathbf{x}_t)]$, we have $\nabla_t^2 \Phi(\mathbf{x}_t) = 0$, which gives the full solution $\mathbf{E}_t(\mathbf{x}_t) = -\nabla_t \Phi(\mathbf{x}_t)$.

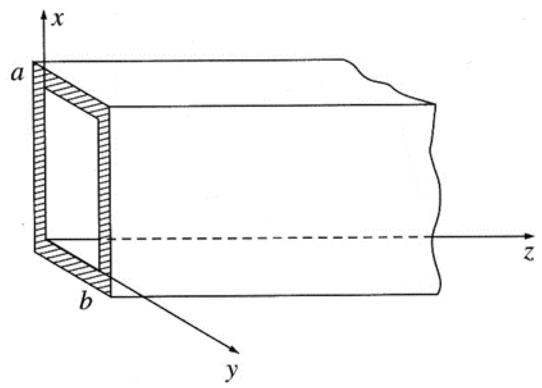
(ii) Note the difference between the scalar potentials discussed here and in Ch. 1 and Ch. 6.

$$\begin{cases} \mathbf{E}_{\text{TEM}} = -\nabla_t \Phi_{\text{TEM}}(\mathbf{x}_t) & \text{regard } \Phi_{\text{TEM}} \text{ as a mathematical tool.} \\ \mathbf{E}(\mathbf{x}) = -\nabla \Phi(\mathbf{x}) & \text{regard } \Phi \text{ as a physical quantity.} \\ \mathbf{E}(\mathbf{x},t) = -\nabla \Phi(\mathbf{x},t) - \frac{\partial}{\partial t} \mathbf{A}(\mathbf{x},t) & \text{regard } \Phi \text{ and } \mathbf{A} \text{ as mathematical tools.} \end{cases}$$

Example 1: TE mode of a rectangular waveguide

Rewrite the basic equations for the TE mode:

$$\begin{cases} (\nabla_t^2 + \gamma^2)H_z = 0 \text{ with boundary condition } \frac{\partial}{\partial n} H_z \Big|_s = 0 & (22) \\ \mathbf{H}_t = \pm \frac{ik_z}{\gamma^2} \nabla_t H_z & (22a) \\ \mathbf{E}_t = \mp \frac{\mu\omega}{k_z} \mathbf{e}_z \times \mathbf{H}_t = \mp Z_h \mathbf{e}_z \times \mathbf{H}_t & (22b) \\ \gamma^2 = \mu\varepsilon\omega^2 - k_z^2 & (22c) \end{cases}$$



Rectangular geometry $\Rightarrow$ Cartesian system $\Rightarrow \nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

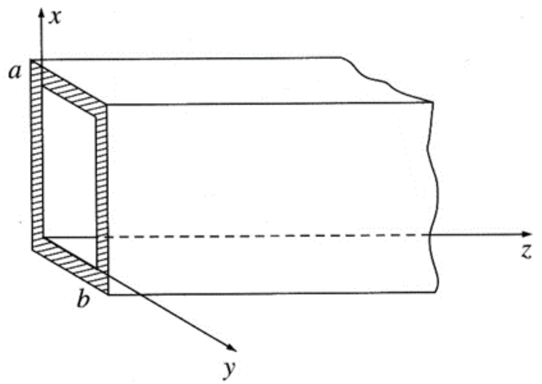
Hence, the wave equation in (22) becomes:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \mu\varepsilon\omega^2 - k_z^2 \right] H_z = 0 \qquad (24)$$

Rewrite (24):
$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\mu\varepsilon\omega^2-k_z^2\right]H_z=0 \qquad (24)$$

Assuming $e^{ik_x x+ik_y y}$ dependence for H_z, we obtain

$$\left[\mu\varepsilon\omega^2-k_x^2-k_y^2-k_z^2\right]H_z=0$$



In order for $H_z\neq 0$, we must have

$$\mu\varepsilon\omega^2-k_x^2-k_y^2-k_z^2=0,$$

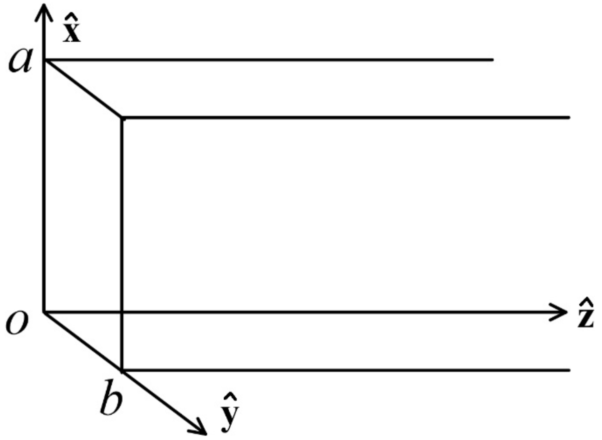
which is satisfied for $\pm k_x,\ \pm k_y,\ \pm k_z$. Since $(e^{ik_x x},\ e^{-ik_x x})$, $(e^{ik_y y},\ e^{-ik_y y})$, and $(e^{ik_z z},\ e^{-ik_z z})$ are all linearly independent pairs, the complete solution for H_z is

$$H_z=e^{-i\omega t}\left[A_1e^{ik_x x}+A_2e^{-ik_x x}\right]\left[B_1e^{ik_y y}+B_2e^{-ik_y y}\right]$$
$$\cdot\left[C_+e^{ik_z z}+C_-e^{-ik_z z}\right] \qquad (25)$$

$$\mathbf{H}_t = \pm \frac{ik_z}{\gamma^2}\nabla_t H_z$$

Applying boundary conditions [see (22)] to (25):

$$H_z = e^{-i\omega t}\left[A_1 e^{ik_x x} + A_2 e^{-ik_x x}\right]\left[B_1 e^{ik_y y} + B_2 e^{-ik_y y}\right]\left[C_+ e^{ik_z z} + C\_ e^{-ik_z z}\right]$$



$$\begin{cases} B_x \propto \frac{\partial}{\partial x} B_z\Big|_{x=0} = 0 \Rightarrow ik_x A_1 - ik_x A_2 = 0 \Rightarrow A_1 = A_2 \\ B_y \propto \frac{\partial}{\partial y} B_z\Big|_{y=0} = 0 \Rightarrow ik_y B_1 - ik_y B_2 = 0 \Rightarrow B_1 = B_2 \end{cases}$$

$$\Rightarrow H_z = \cos k_x x \cos k_y y\left[C_+ e^{-i\omega t + ik_z z} + C\_ e^{-i\omega t - ik_z z}\right]$$

$$\begin{cases} B_x \propto \frac{\partial}{\partial x} B_z\Big|_{x=a} = 0 \Rightarrow \sin k_x a = 0 \Rightarrow k_x = m\pi/a,\ m = 0,1,2,\ldots \\ B_y \propto \frac{\partial}{\partial y} B_z\Big|_{y=b} = 0 \Rightarrow \sin k_y b = 0 \Rightarrow k_y = n\pi/b,\ n = 0,1,2,\ldots \end{cases}$$

$$\Rightarrow H_z = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\Big[\underbrace{C_+ e^{ik_z z - i\omega t}}_{\text{forward wave}} + \underbrace{C\_ e^{-ik_z z - i\omega t}}_{\text{backward wave}}\Big] \qquad (26)$$

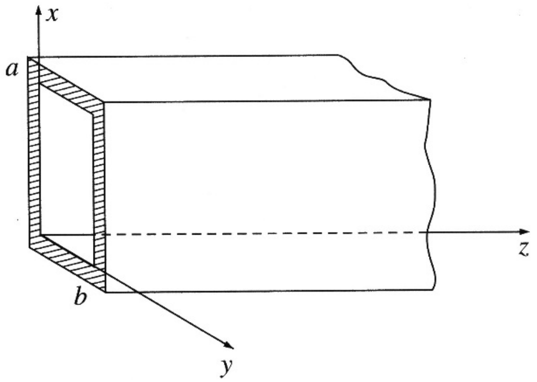
Substituting $k_x = \frac{m\pi}{a}$, $k_y = \frac{n\pi}{b}$ into $\mu\varepsilon\omega^2 - k_x^2 - k_y^2 - k_z^2 = 0$, we obtain

$$\mu\varepsilon\omega^2 - k_z^2 - \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) = 0,\ m,n = 0,1,2,\ldots \qquad (27)$$

Griffiths

9.5.2 TE Waves in a Rectangular Wave Guide

$E_z = 0$, and $B_z(x, y) = X(x)Y(y) \leftarrow$ separation of variables



$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \left(\frac{\omega^2}{v^2} - k^2\right) = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \quad \text{and} \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2$$

with $\frac{\omega^2}{v^2} = k^2 + k_x^2 + k_y^2$

*Griffiths' derivation uses different boundary condition --- $\mathbf{E}_{||}=0$.

$$X(x) = A\sin k_x x + B\cos k_x x$$
$$Y(y) = C\sin k_y y + D\cos k_y y$$

Griffiths

TE Waves in a Rectangular Wave Guide (II)

$$\mathbf{E}_t = \mp \frac{i\mu\omega}{\gamma^2} \mathbf{e}_z \times \nabla_t H_z$$

$$E_x \propto \frac{\partial B_z}{\partial y} \propto C\cos k_y y - D\sin k_y y$$

$$E_x(@\, y = 0) = 0 \Rightarrow C = 0$$

$$E_x(@\, y = b) = 0 \Rightarrow \sin k_y b = 0, k_y = \frac{n\pi}{b} (n = 0,1,2,...)$$

$$E_y \propto \frac{\partial B_z}{\partial x} \propto A\cos k_x x - B\sin k_x x$$

$$E_y(@\, x = 0) = 0 \Rightarrow A = 0$$

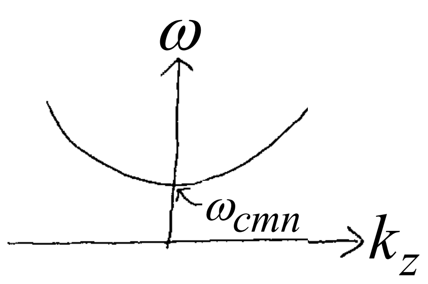
$$E_y(@\, x = a) = 0 \Rightarrow \sin k_x a = 0, k_x = \frac{m\pi}{a} (m = 0,1,2,...)$$

$$B_z(x, y) = B_0 \cos(m\pi x / a)\cos(n\pi y / b) \leftarrow \text{the TE}_{\text{mn}} \text{ mode}$$

$$k = \sqrt{(\omega / v)^2 - \pi^2[(m / a)^2 + (n / b)^2]}$$

Rewrite (27) as $\mu\varepsilon\omega^2 - k_z^2 - \mu\varepsilon\omega_{cmn}^2 = 0,$ (28)

where $\omega_{cmn} = \frac{\pi}{\sqrt{\mu\varepsilon}}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2}, \quad m, n = 0, 1, 2, \ldots$ (29)



Each pair of (m, n) gives a normal mode ($\mathrm{TE_{mn}}$ mode) of the waveguide. m and n cannot both be 0, because that will creat a $\frac{0}{0}$ situation on (8.26) or (22a), making $\mathbf{H}_t$ and $\mathbf{E}_t$ indeterminable.

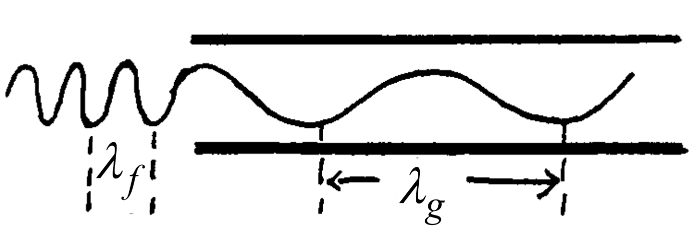
ω_{cmn} is the cutoff frequency (the frequency at which $k_z = 0$) of the waveduide for the $\mathrm{TE_{mn}}$ mode. Waves with $\omega < \omega_{cmn}$ cannot propagate as a $\mathrm{TE_{mn}}$ mode because k_z becomes purely imaginary.

(28) is the $\mathrm{TE_{mn}}$ mode dispersion relation of a waveguide filled with a dielectric medium with constant (in general complex) ε and μ.

For the usual case of an unfilled waveguide, we have $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ ($\Rightarrow \mu\varepsilon = \mu_0\varepsilon_0 = \frac{1}{c^2}$), and (28) (29) can be written

$$\omega^2 - k_z^2 c^2 - \omega_{cmn}^2 = 0 \text{ with } \omega_{cmn} = \pi c\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2} \quad \left[\begin{array}{l}\text{for unfilled}\\ \text{waveguide}\end{array}\right] \quad (30)$$

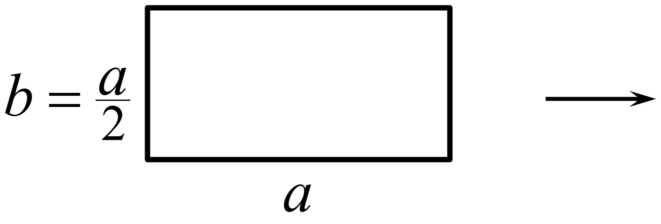
$$\omega^2 - k_z^2 c^2 - \omega_{cmn}^2 = 0, \ \ \omega_{cmn} = \pi c \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^{\frac{1}{2}}$$

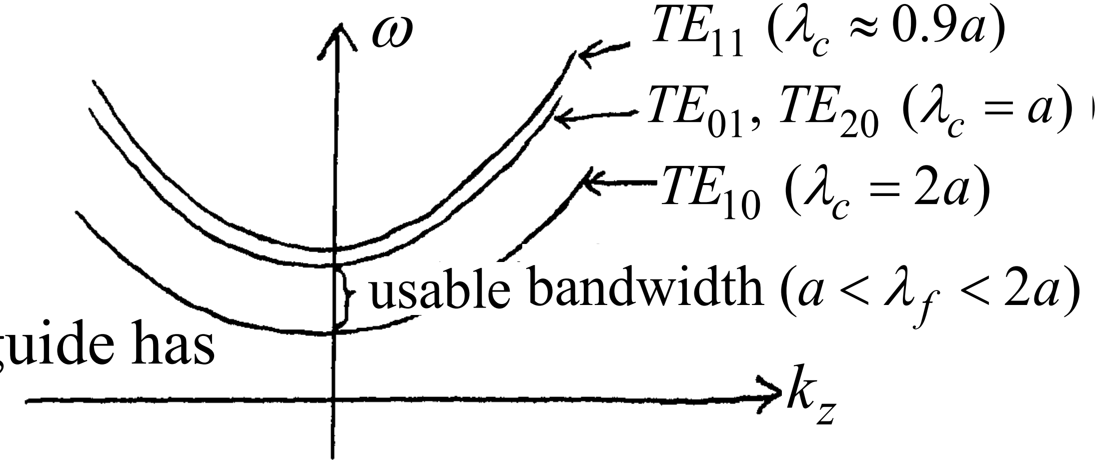


$$\begin{cases} \lambda_g = \underline{\text{guide wavelength}} \equiv 2\pi / k_z \\ \lambda_c = \underline{\text{cutoff wavelength}} \equiv 2\pi c / \omega_{cmn} \\ \lambda_f = \underline{\text{free space wavelength}} \equiv 2\pi c / \omega \end{cases}$$

depend on →	mode & geometry	wave freq.
λ_g	yes	yes
λ_c	yes	no
λ_f	no	yes

$$\begin{cases} \omega > \omega_{cmn} \Rightarrow k_z = \text{real} \Rightarrow \underline{\text{propagating waves}} \\ \omega = \omega_{cmn} \Rightarrow k_z = 0 \Rightarrow \lambda_g = \infty \\ \omega < \omega_{cmn} \Rightarrow k_z = \text{imaginary} \Rightarrow \underline{\text{evanescent fields}} \end{cases}$$



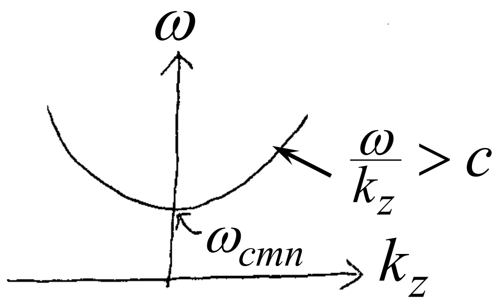


Question 1: A typical waveguide has $a = 2b$. Why?

Question 2: Can we use a waveguide to transport waves at 60 Hz?

Other quantities of interest:

(1) Differentiating $\omega^2 - k_z^2 c^2 - \omega_{cmn}^2 = 0$ with respect to k_z



$$2\omega \frac{d\omega}{dk_z} - 2k_z c^2 = 0$$

$$\Rightarrow \; v_g = \frac{d\omega}{dk_z} = \frac{k_z c^2}{\omega} \quad \text{[group velocity in unfilled waveguide]}$$

$$\Rightarrow \begin{cases} v_g < c \\ v_g \to 0 \text{ as } \omega \to \omega_{\text{cmn}} \end{cases}$$

(2) The remaining field components (E_x, E_y, H_x, and H_y) can be obtained from H_z through

$$\mathbf{H}_t = \pm \frac{ik_z}{\gamma^2} \nabla_t H_z \qquad (22a)$$

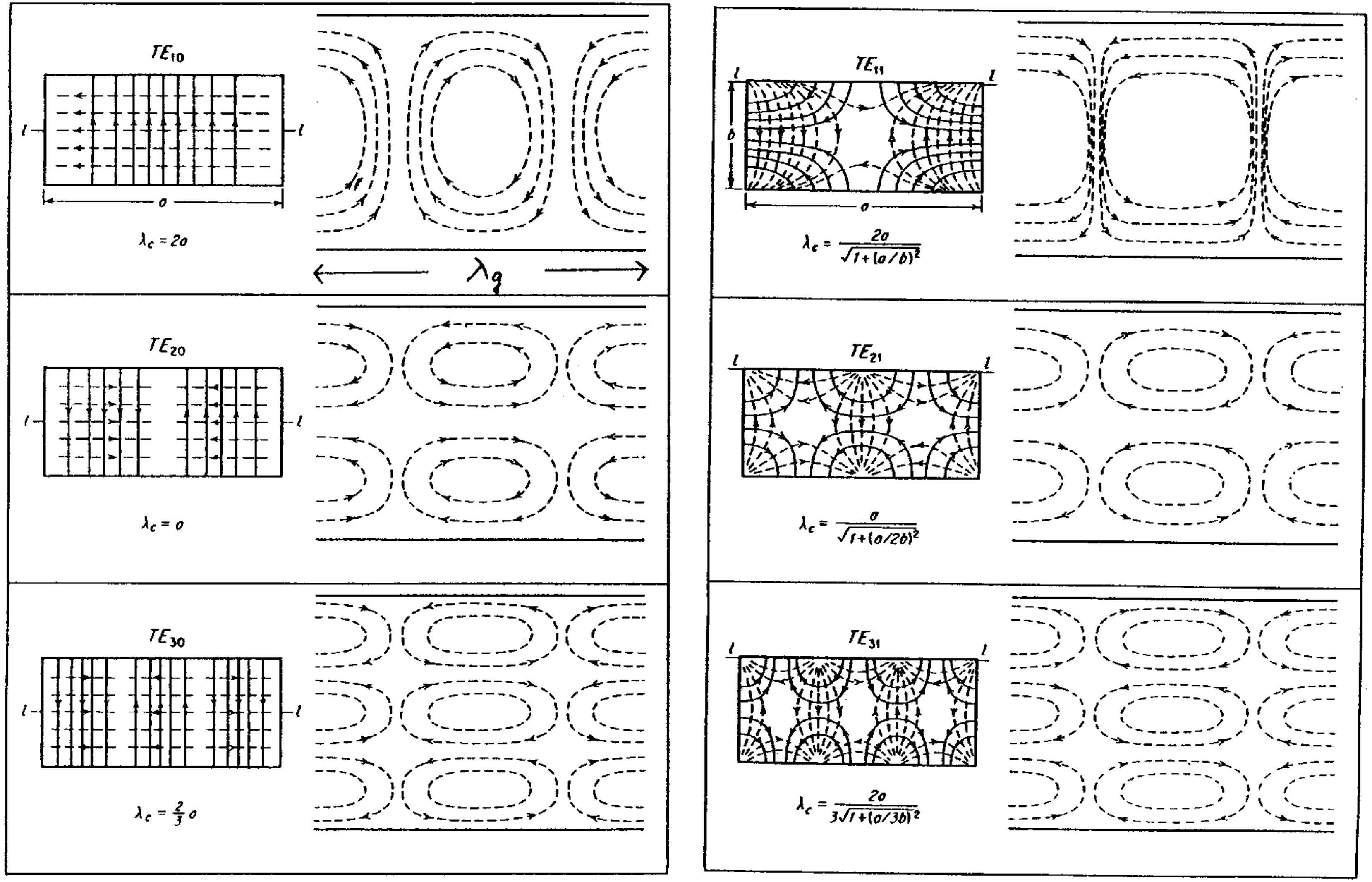
$$\mathbf{E}_t = \mp \frac{\mu\omega}{k_z} \mathbf{e}_z \times \mathbf{H}_t \qquad (22b)$$

[$\gamma^2 = \mu\varepsilon\omega^2 - k_z^2 = \frac{\omega_{cmn}^2}{c^2}$ see (22c) and (30).]

where the $\begin{Bmatrix} \text{upper} \\ \text{lower} \end{Bmatrix}$ sign applies to the $\begin{Bmatrix} \text{forward} \\ \text{backward} \end{Bmatrix}$ wave.

8.2-8.4 Modes in Waveguides (*continued*)

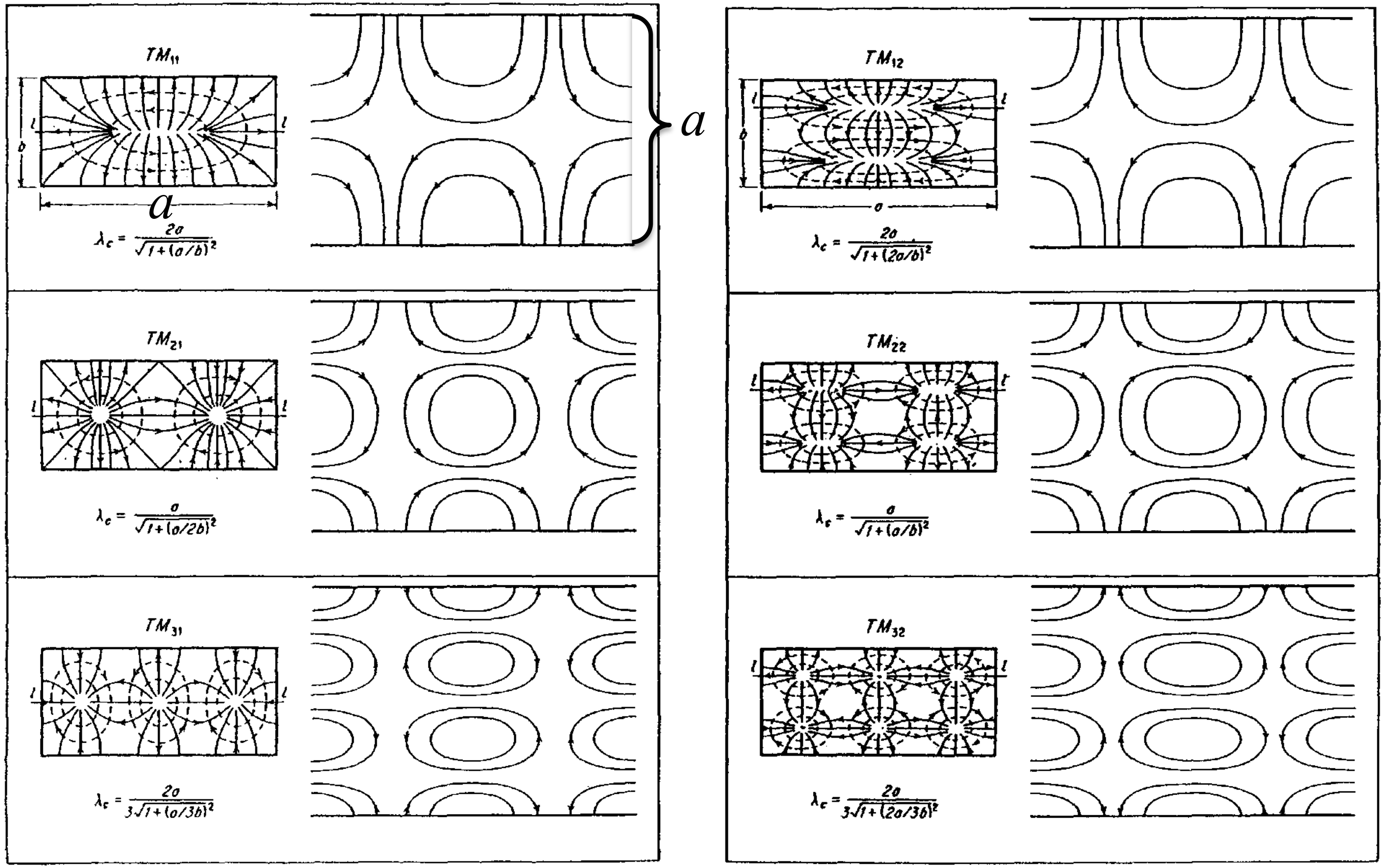
TE mode field patterns of rectangular waveguide



from E. L. Ginzton, "Microwave measurements". λ_c : cutoff frequency

solid curve: **E**-field lines; dashed curves: **B**-field lines

TM mode field patterns of rectangular waveguide



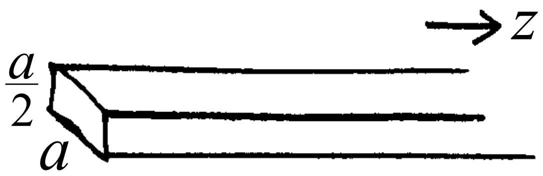
from E. L. Ginzton, "Microwave measurements". λ_c : cutoff frequency
solid curve: **E**-field lines; dashed curves: **B**-field lines

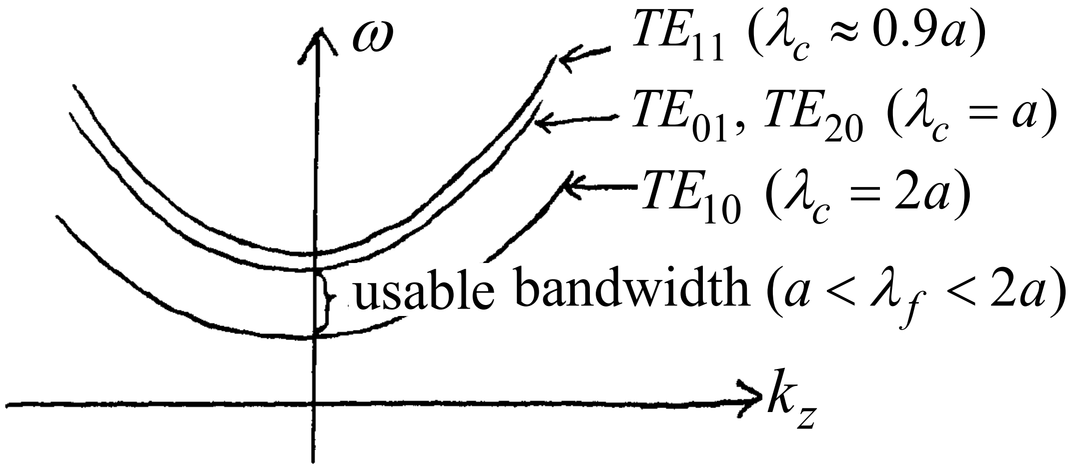
8.2-8.4 Modes in Waveguides (*continued*)

Discussion: Waveguide and microwaves

A typical waveguide has $a = 2b$ to maximize the usable bandwidth ($a < \lambda_f < 2a$) over which only the TE_{10} mode can propagate and hence mode purity is maintained. Waves are normally transported by the TE_{10} mode over this frequency range. Waveguides come in different sizes. Usable bandwidths of waveguides of practical dimensions (0.1 cm < a < 100 cm) cover the entire microwave band (300 MHz to 300 GHz).

Compared with coaxial transmission lines, the waveguide is capable of handling much higher power. Hence, it is commonly used in high-power microwave systems. In a radar system, for example, it is used to transport microwaves from the generator to the antenna.

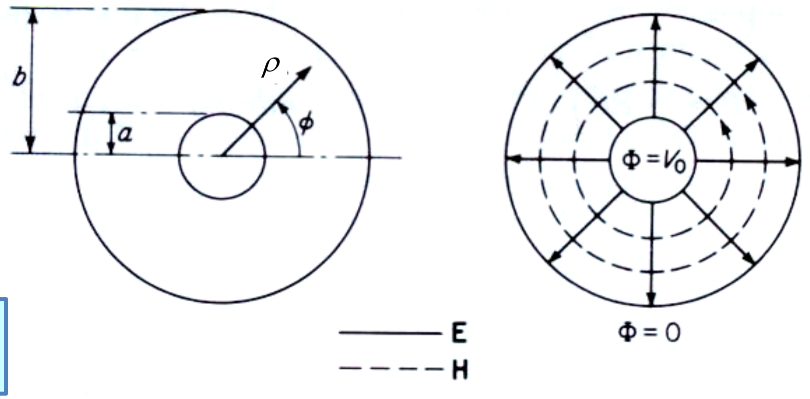




Example 2: TEM modes of a coaxial transmission line

TEM modes are governed by the following set of equtions:

$$\begin{cases} \nabla_t^2 \Phi_{\text{TEM}}(\mathbf{x}_t) = 0 & (23) \\ \mathbf{E}_{\text{TEM}} = -\nabla_t \Phi_{\text{TEM}}(\mathbf{x}_t) & (23a) \\ \mathbf{H}_{\text{TEM}} = \pm \frac{1}{Z} \mathbf{e}_z \times \mathbf{E}_{\text{TEM}} & (23b) \\ \omega^2 = \frac{k_z^2}{\mu\varepsilon} \qquad Z = \sqrt{\mu/\varepsilon} & (23c) \end{cases}$$



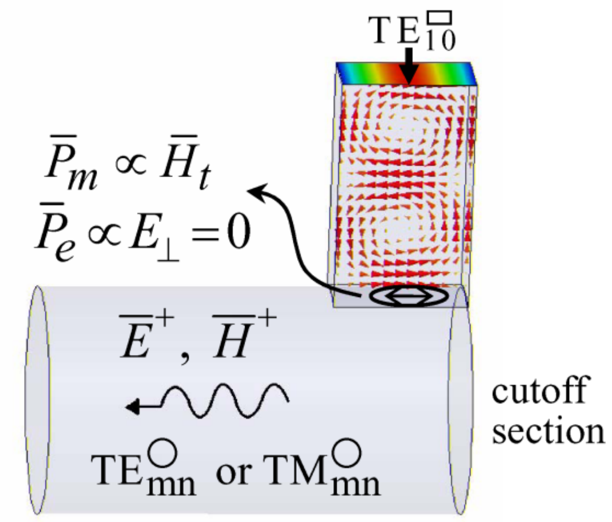
(23) gives $\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\Phi_{\text{TEM}}}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2\Phi_{\text{TEM}}}{\partial\phi^2} = 0.$

Neglect the $\frac{\partial\Phi}{\partial\phi} \neq 0$ modes $\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial\Phi_{\text{TEM}}}{\partial\rho}) = 0 \Rightarrow \Phi_{\text{TEM}} = C_1 ln(\rho) + C_2.$

Apply b.c. $\begin{cases} \Phi_{\text{TEM}}(\rho = a) = V_0 \\ \Phi_{\text{TEM}}(\rho = b) = 0 \end{cases} \Rightarrow \begin{cases} C_1 = V_0 / ln(a/b) \\ C_2 = -C_1 ln(b) \end{cases} \Rightarrow \Phi_{\text{TEM}} = V_0 \frac{ln(\rho/b)}{ln(a/b)}.$

(23a, b) then give
$$\begin{cases} \mathbf{E}_{\text{TEM}}(\mathbf{x},t) = \frac{V_0}{ln(b/a)}\frac{1}{\rho} e^{\pm ik_z z - i\omega t}\mathbf{e}_\rho \\ \mathbf{H}_{\text{TEM}}(\mathbf{x},t) = \pm\frac{V_0}{Zln(b/a)}\frac{1}{\rho} e^{\pm ik_z z - i\omega t}\mathbf{e}_\phi \end{cases} \quad (31)$$

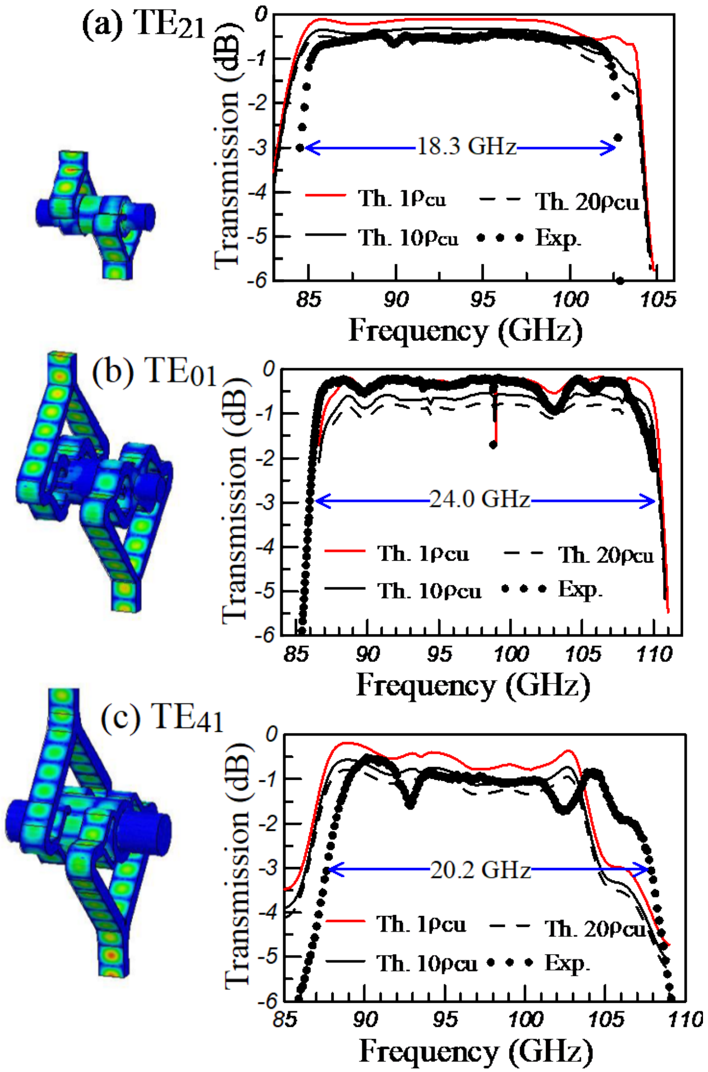
Exciting a Specific Mode



$$\bar{M} = j\omega\mu_0 \bar{P}_m$$

$$\bar{E}_2^+ = \sum_n A_n^+ \left(\bar{e}_n + \hat{z} e_{zn}\right) \cdot e^{-j\beta_n z}$$

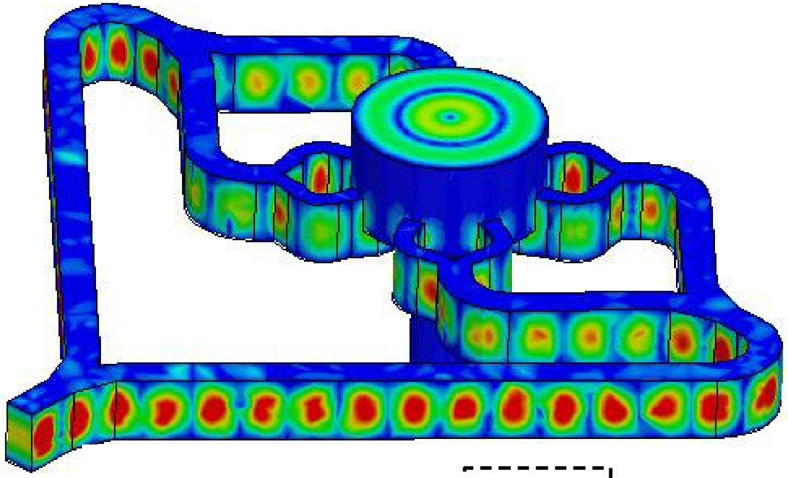
$$\frac{P_{01}^{total}}{P_{41}^{total}} = \frac{\beta_{41} {p'_{01}}^4 \varepsilon_{04}}{\beta_{01} {p'_{41}}^4 \varepsilon_{00}} \frac{\left({p'_{01}}^2 - 0^2\right)}{\left({p'_{41}}^2 - 4^2\right)} \frac{J_0^4(p'_{01})}{J_4^4(p'_{41})} = -18 \text{ dB}$$

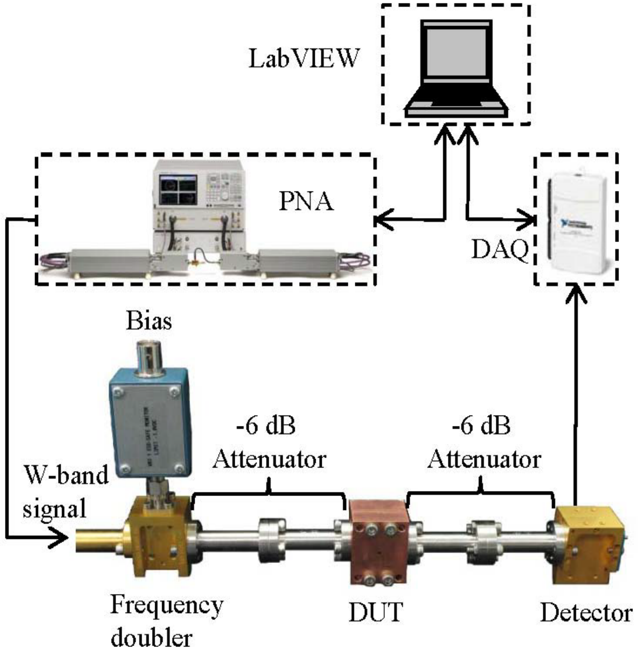


Appl. Phys. Lett. **93**, 111503 (2008)

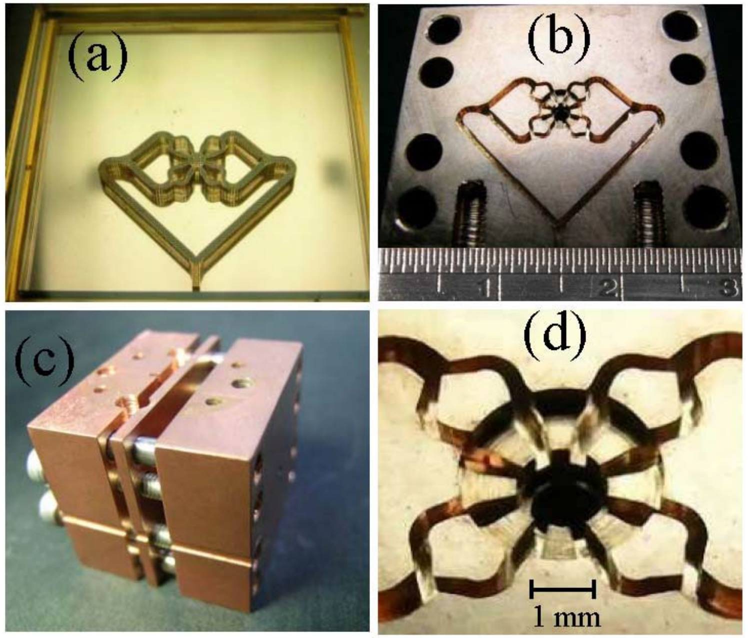
IEEE Trans. Microwave Theory Tech. **58**, 1543 (2010)

Difficulties of Exciting a Higher-Order Mode: Take TE_{02} as an Example





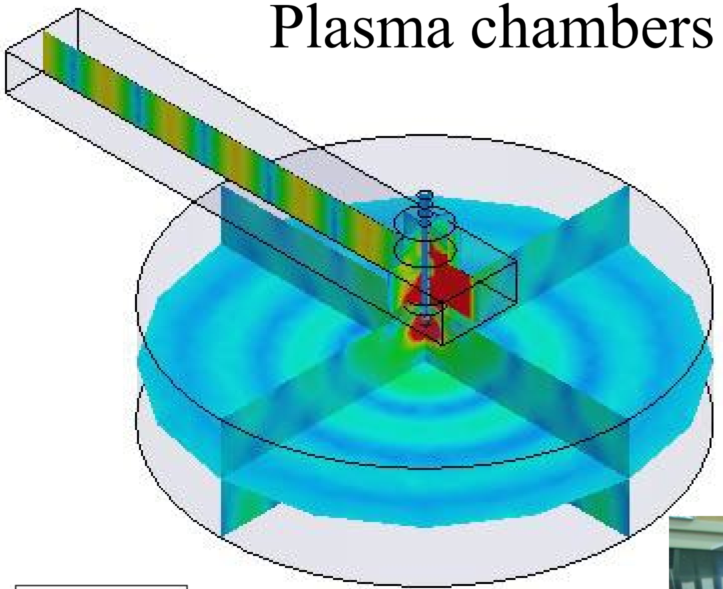
Rev. Sci. Instrum. **81**, 054701 (2010).

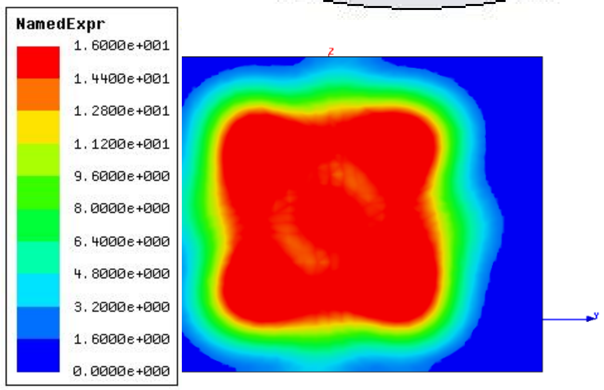


Desired mode	TE_{02}	
Coupling structure	octa-feed	
Waveguide radius	1.86 mm	
Parasitic modes	$TE_{11,A}$, $TE_{11,B}$	$TE_{21,A}$, $TE_{21,B}$
	TE_{01}	
	$TE_{31,A}$, $TE_{31,B}$	$TE_{12,A}$, $TE_{12,B}$
	TM_{01}	
	$TM_{11,A}$, $TM_{11,B}$	$TM_{21,A}$, $TM_{21,B}$
	$TE_{41,A}$, $TE_{41,B}$	$TE_{12,A}$, $TE_{12,B}$
	TM_{02}	
	$TM_{31,A}$, $TM_{31,B}$	$TE_{51,A}$, $TE_{51,B}$
	$TE_{22,A}$, $TE_{22,B}$	$TM_{12,A}$, $TM_{12,B}$

Applications of Waveguide Modes (I)

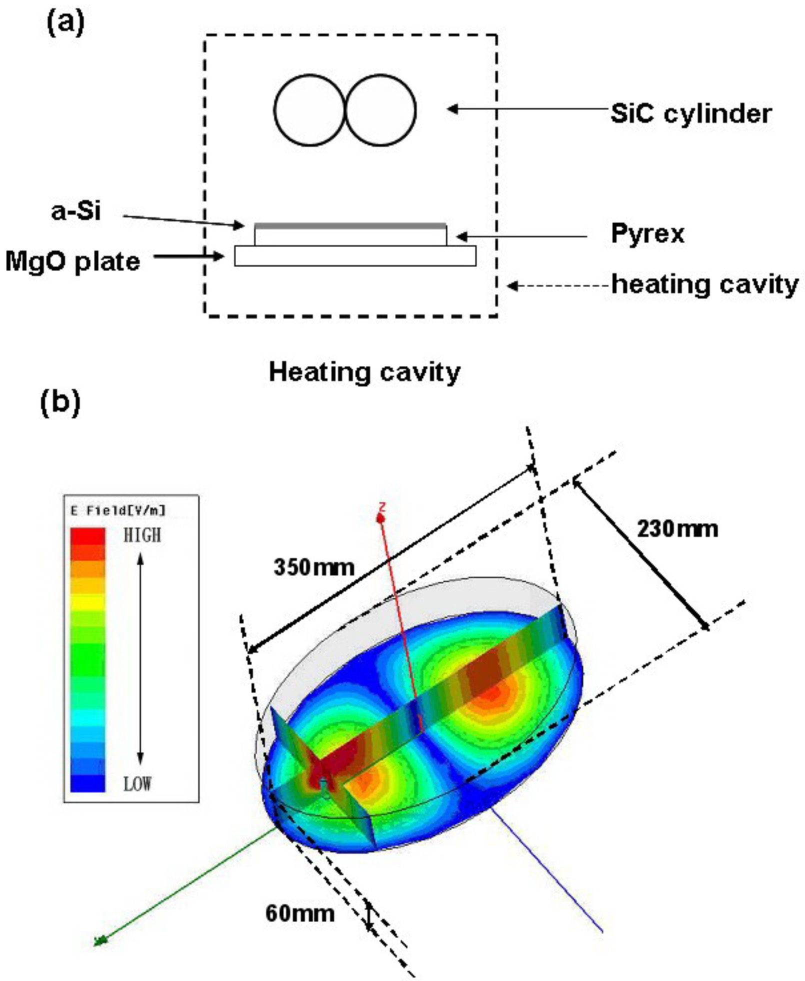
Plasma chambers





Phys. Plasmas **19**, 033302 (2012)
Appl. Phys. Lett. **101**, 062414 (2012)

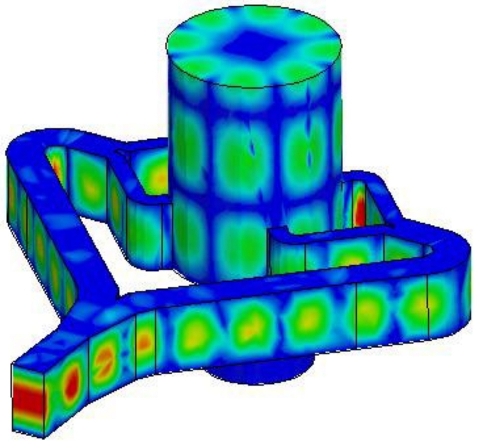
Material processing

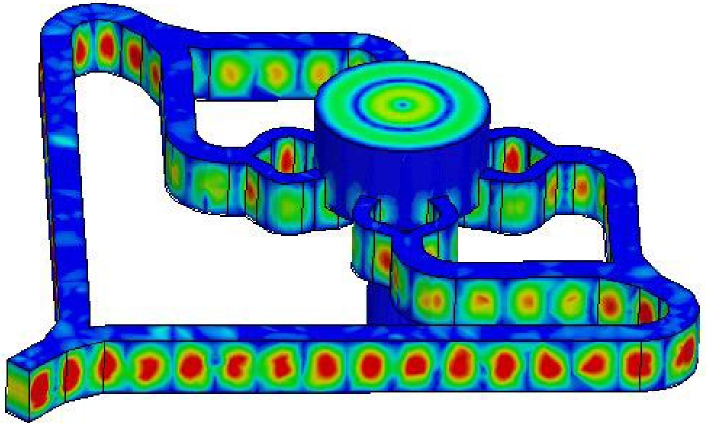


Appl. Phys. Lett. **94**, 102104 (2009)

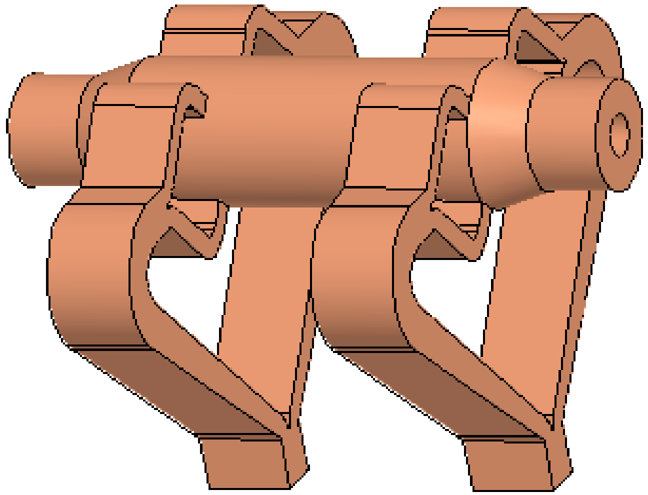
Applications of Waveguide Modes (II)

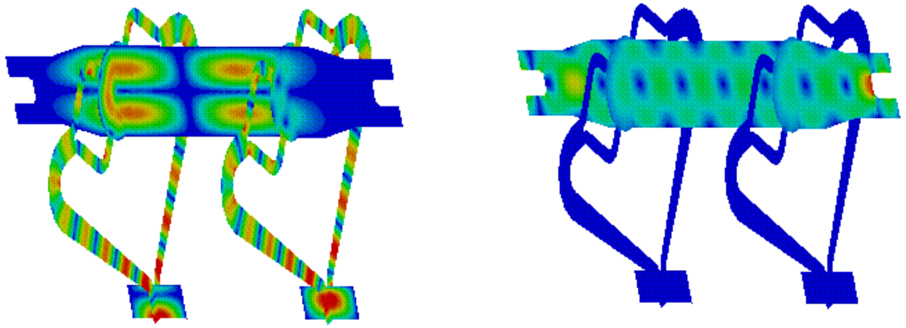
Mode converters





Rotary joints





Rev. Sci. Instrum. 80, 034701 (2009).

THz waveguide, circulator, isolator, power divider, antenna…

8.5 Energy Flow and Attenuation in Waveguides

Power in a Lossless Waveguide: Consider a TM mode ($\mathbf{E} = \mathbf{E}_t + E_z \mathbf{e}_z$, $\mathbf{H} = \mathbf{H}_t$) in a medium with real ε, μ (hence real ω, k_z).

$$\mathbf{S}_{\text{TM}} = \tfrac{1}{2}\mathbf{E}\times\mathbf{H}^* = \tfrac{1}{2}[\mathbf{E}_t \times \mathbf{H}_t^* + E_z \mathbf{e}_z \times \mathbf{H}_t^*] \quad \text{[complex Poynting vector]}$$

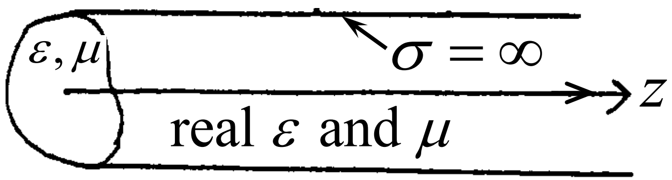
$$= \frac{1}{2}\frac{\varepsilon\omega}{k_z}[\underbrace{\mathbf{E}_t \times (\mathbf{e}_z \times \mathbf{E}_t^*)}_{\mathbf{e}_z|\mathbf{E}_t|^2} + E_z \underbrace{\mathbf{e}_z \times (\mathbf{e}_z \times \mathbf{E}_t^*)}_{-\mathbf{E}_t^*}] \quad \text{[for TM modes]}$$

(21b)

$$= \frac{\varepsilon\omega}{2k_z}[\mathbf{e}_z \frac{k_z^2}{\gamma^4}|\nabla_t E_z|^2 + \frac{ik_z}{\gamma^2}E_z \nabla_t E_z^*]$$

(21a)

$$= \frac{\omega k_z \varepsilon}{2\gamma^4}[\mathbf{e}_z|\nabla_t E_z|^2 + \frac{i\gamma^2}{k_z}E_z \nabla_t E_z^*]$$



The complex Poynting theorem:

$$\tfrac{1}{2}\int_v \mathbf{J}^* \cdot \mathbf{E} d^3x + 2i\omega \int_v (w_e - w_m) d^3x + \oint_s \mathbf{S}\cdot\mathbf{n} da = 0, \qquad (6.134)$$

where $\mathbf{S} \equiv \tfrac{1}{2}\mathbf{E}\times\mathbf{H}^*$ [called the complex Poynting vector] (6.132)

P_{TM} = time averaged power in the z-direction

$$= \int_A \mathbf{e}_z \cdot [\text{Re}\,\mathbf{S}_{\text{TM}}] da \quad [A\text{: crossectional area}]$$

$$= \frac{\omega k_z \varepsilon}{2\gamma^4}\int_A (\nabla_t E_z^* \cdot \nabla_t E_z) da \qquad (35)$$

Green's first identity: $\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)d^3x = \oint_S \phi\frac{\partial\psi}{\partial n}da$ (1.34)

Let ϕ and ψ be independent of z and apply (1.34) to a slab of end surface area A (on the x-y plane) and infinistesimal thickness Δz in z,

$$\Delta z\int_A (\phi\nabla_t^2\psi + \nabla_t\phi\cdot\nabla_t\psi)da = \Delta z\oint_C \phi\frac{\partial\psi}{\partial n}dl + \left[\begin{array}{l}\text{surface integrals on}\\ \text{two ends of the}\\ \text{slab, which vanish.}\end{array}\right]$$

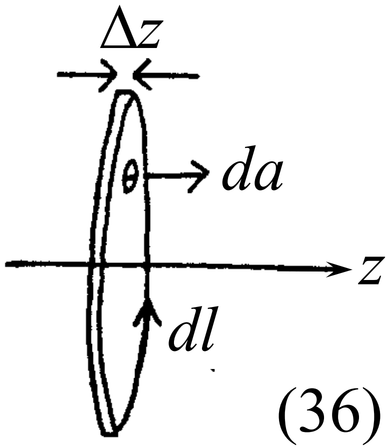
$$\Rightarrow \int_A (\phi\nabla_t^2\psi + \nabla_t\phi\cdot\nabla_t\psi)da = \oint_C \phi\frac{\partial\psi}{\partial n}dl$$

Let $\phi = E_z^*$ and $\psi = E_z$, then

$$\int_A (\nabla_t E_z^*\cdot\nabla_t E_z)da = [\oint_C \underbrace{E_z^*}_{=0}\frac{\partial}{\partial n}E_z dl - \int_A E_z^*\underbrace{\nabla_t^2 E_z}_{-\gamma^2 E_z} da]$$

by boundary condition; by (14)

$$= \gamma^2\int_A |E_z|^2 da. \quad (36)$$



Sub. (36) into (35): $P_{\text{TM}} = \frac{\omega k_z\varepsilon}{2\gamma^4}\int_A (\nabla_t E_z^*\cdot\nabla_t E_z)da$, we obtain

$$P_{\text{TM}} = \frac{\omega k_z\varepsilon}{2\gamma^2}\int_A |E_z|^2 da, \quad [\text{where } \gamma^2 = \mu\varepsilon\omega^2 - k_z^2] \quad (37)$$

$$\gamma^2 = \mu\varepsilon\omega^2 - k_z^2 \;\Rightarrow \omega_c = \frac{\gamma}{\sqrt{\mu\varepsilon}} \quad \left[\begin{array}{l}\omega_c \text{(i.e., } \omega \text{ at } k_z = 0\text{) is the} \\ \text{cutoff freq. of the mode.}\end{array}\right] \tag{38}$$

$$\Rightarrow \; k_z = (\mu\varepsilon\omega^2 - \gamma^2)^{\frac{1}{2}} = \sqrt{\mu\varepsilon}\,\omega(1 - \frac{\omega_c^2}{\omega^2})^{\frac{1}{2}} \tag{39}$$

Substituting (38) and (39) into (37)

$$P_{\text{TM}} = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}(\frac{\omega}{\omega_c})^2(1 - \frac{\omega_c^2}{\omega^2})^{\frac{1}{2}}\int_A |E_z|^2\,da \qquad \text{[cf. (8.51)]} \tag{40}$$

Similarly, for the TE mode and real μ, ε, ω, and k_z, we obtain from (22), (22a), and (22b),

$$\mathbf{S}_{\text{TE}} = \frac{\omega k_z \mu}{2\gamma^4}[\mathbf{e}_z |\nabla_t H_z|^2 - \frac{i\gamma^2}{k_z} H_z^* \nabla_t H_z] \tag{41}$$

$$P_{\text{TE}} = \int_A \mathbf{e}_z \cdot [\text{Re}\,\mathbf{S}_{\text{TE}}]da = \frac{\omega k_z \mu}{2\gamma^2}\int_A |H_z|^2\,da$$

$$= \frac{1}{2}\sqrt{\frac{\mu}{\varepsilon}}(\frac{\omega}{\omega_c})^2(1 - \frac{\omega_c^2}{\omega^2})^{\frac{1}{2}}\int_A |H_z|^2\,da \qquad \text{[cf. (8.51)]} \tag{42}$$

Note: P_{TM} and P_{TE} are expressed in terms of the generating function.

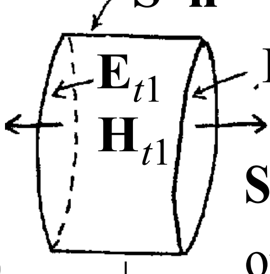
Energy in a Lossless Waveguide:

$$\oint_S \mathbf{S}\cdot\mathbf{n}da + \tfrac{1}{2}\int_V \mathbf{J}^*\cdot\mathbf{E}d^3x + 2i\omega\int_V (w_e - w_m)d^3x = 0 \quad (6.134)$$

$$\begin{cases} w_e = \frac{1}{4}\mathbf{E}\cdot\mathbf{D}^* = \frac{1}{4}\varepsilon|E|^2 \\ w_m = \frac{1}{4}\mathbf{B}\cdot\mathbf{H}^* = \frac{1}{4\mu}|B|^2 \end{cases} \quad (6.133)$$

[if ε, μ are real, w_e and w_m are also real and represent time averaged field energy densities.]

Apply (6.134) to a section of a lossless waveguide [i.e., μ, ε are real and the wall conductivity $\sigma = \infty$].



$\sigma = 0$ (inside volume) $\Rightarrow \mathbf{J} = 0 \Rightarrow \int_V \mathbf{J}^*\cdot\mathbf{E}d^3x = 0$

$\mathbf{E}_t = 0$ on the side wall $\Rightarrow \mathbf{S}\cdot\mathbf{n} = 0$ on the side wall

μ, ε (hence ω, k_z) are real $\Rightarrow \mathbf{E}_t$ and $\mathbf{H}_t$ are in phase [by (21b)&(22b)]

$\Rightarrow \mathbf{E}_t \times \mathbf{H}_t^*$ is real $\Rightarrow \mathbf{S}$ is real on both ends $\Rightarrow \oint_S \mathbf{S}\cdot\mathbf{n}da$ is real

$$\begin{cases} \text{Re}[(6.134)] \Rightarrow \oint_S \mathbf{S}\cdot\mathbf{n}da = 0 \quad \text{(no net power into or out of volume)} \\ \text{Im}[(6.134)] \Rightarrow \int_V w_e d^3x = \int_V w_m d^3x \quad \text{(B-field energy = E-field energy)} \end{cases}$$

For the TM mode ($H_z = 0$):

U_{TM} = field energy per unit length

$$= \int_A (w_e + w_m) da = 2\int_A w_m da = \frac{\mu}{2}\int_A |\mathbf{H}_t|^2 da \overset{(21b)}{=} \frac{\mu}{2}\frac{\varepsilon^2\omega^2}{k_z^2}\int_A |\mathbf{E}_t|^2 da$$

$$\overset{(21a)}{=} \frac{\mu}{2}\frac{\varepsilon^2\omega^2}{k_z^2}\frac{k_z^2}{\gamma^4}\underbrace{\int_A |\nabla_t E_z|^2 da}_{\gamma^2\int_A |E_z|^2 da \text{ by (36)}} = \frac{\mu\varepsilon^2\omega^2}{2\gamma^2}\int_A |E_z|^2 da \overset{\gamma^2 = \mu\varepsilon\omega_c^2}{=} \frac{\varepsilon}{2}\left(\frac{\omega}{\omega_c}\right)^2 \int_A |E_z|^2 da \quad (43)$$

Similarly, for the TE mode ($E_z = 0$):

$$U_{\text{TE}} = 2\int_A W_e da = \frac{\varepsilon}{2}\int_A |\mathbf{E}_t|^2 da = \frac{\mu}{2}\left(\frac{\omega}{\omega_c}\right)^2 \int_A |H_z|^2 da \quad (44)$$

Use (22a,b) and Green's 1st identity

From (40), (42), (43), and (44)

$$\frac{P_{\text{TM}}}{U_{\text{TM}}} = \frac{P_{\text{TE}}}{U_{\text{TE}}} = \frac{1}{\sqrt{\mu\varepsilon}}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{\frac{1}{2}} \overset{(39)}{=} \frac{k_z}{\mu\varepsilon\omega} \overset{\frac{d}{dk_z}(21.c)}{=} v_g \quad (8.53)$$

$$v_p = \omega / k_z$$

$$\Rightarrow v_p v_g = 1/\mu\varepsilon \quad (8.54)$$

Attenuation in Waveguides Due to Ohmic Loss on the Wall:

We express k_z for a lossless ($\sigma = \infty$) and lossy ($\sigma \neq \infty$) waveguide

as $$k_z = \begin{cases} k_z^{(0)}, & \sigma = \infty \\ k_z^{(0)} + \alpha + i\beta, & \sigma \neq \infty \end{cases} \quad (8.55)$$

where $k_z^{(0)}$ is the solution of the dispersion relation for $\sigma = \infty$, i.e.,

$$\mu\varepsilon\omega^2 - k_z^2 - \mu\varepsilon\omega_c^2 = 0 \quad \text{[derived in (28)]} \quad (45)$$

The expression for $\sigma \neq \infty$ in (8.55) assumes that the wall loss modifies $k_z^{(0)}$ by a small real part α and a small imaginary part β, where α and β are to be dertermined.

Physical reason for α: Effective waveguide radius increases by by an amount ~ skin depth δ. A larger waveguide has a smaller ω_c. Hence, $\alpha > 0$.

Physical reason for β: Power dissipation on the wall.

In $k_z = k_z^{(0)} + \alpha + i\beta$, α is not of primary interest because it modifies the guide wavelength slightly. However, β results in attenuation, which can be very significant over a long distance. We outline below how β can be evaluated.

$$P = \text{power flow} \left(\propto \text{Re}[\mathbf{E}_t \times \mathbf{H}_t^*] \propto e^{ik_z z} \cdot e^{-ik_z^* z} = e^{-2k_{zi} z} = e^{-2\beta z} \right)$$

$$= P_0 e^{-2\beta z} \qquad (8.56)$$

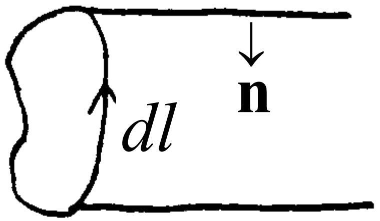
power dissipation/unit length

$$\Rightarrow \quad \beta = -\frac{1}{2P}\frac{dP}{dz} = \text{field attenuation constant} \qquad (8.57)$$

$$\frac{dP_{loss}}{da} = \frac{1}{2\sigma\delta}\left|\mathbf{K}_{eff}\right|^2 \quad (8.15)$$

$$(8.15) \Rightarrow \frac{dP}{dz} = -\frac{1}{2\sigma\delta}\oint_c \left|\mathbf{K}_{eff}\right|^2 dl \qquad (46)$$

$$(8.14) \Rightarrow \mathbf{K}_{eff} = \mathbf{n} \times \mathbf{H} \qquad (47)$$



$$(46)(47) \Rightarrow \frac{dP}{dz} = -\frac{1}{2\sigma\delta}\oint_c \left|\mathbf{n} \times \mathbf{H}\right|^2 dl \qquad (8.58)$$

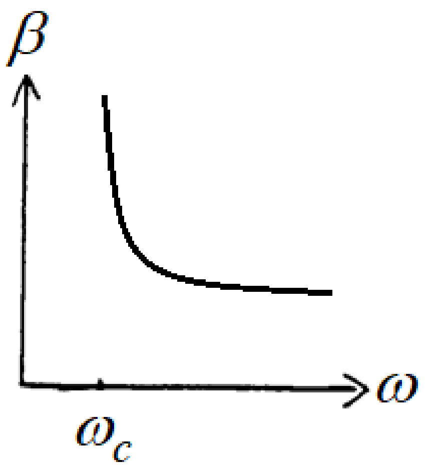
Since the wall loss can be regarded as a small perturbation, we may use the zero-order $\mathbf{H}$ derived for $\sigma = \infty$ in Sec.8.1 to calculate $\frac{dP}{dz}$.

Specifically, we calculate the zero-order **E** and **H,** and use the zero-order **E** and **H** to calculate P from (8.51) and dP/dz from (8.58). β is then found from (8.57).

Formulae for β for rectangular and cylindrical waveguides are tabulated in many microwave textbooks, e.g., R. E. Collin, "Foundation of Microwave Engineering" (2nd Ed.) p. 189 & p.197 (where the attenuation constant is denoted by α instead of β).

Note:

(i) β has been calculated by a perturbation method. The method is invalid near the cutoff frequency, at which there is a large "perturbation". Sec 8.6 gives a method which calculates both α and β (due to wall loss) valid for all frequencies.



(ii) Other types of losses (e.g., lossy filling medium or complex ε) can also contribute to α and β.

(iii) Note there are two definitions of the attenuation constant.

In Ch. 8 of Jackson, the attentuation constant for the waveguide is denoted by β and it is defined as

$$\beta = -\frac{1}{2P}\frac{dP}{dz}, \tag{8.57}$$

This is the *field* attentuation constant, i.e.,

$$\mathbf{E}, \mathbf{B} \propto e^{-\beta z}.$$

In Ch. 7 of Jackson, the attentuation constant for a uniform medium is denoted by α [see (7.53)] and it is defined as

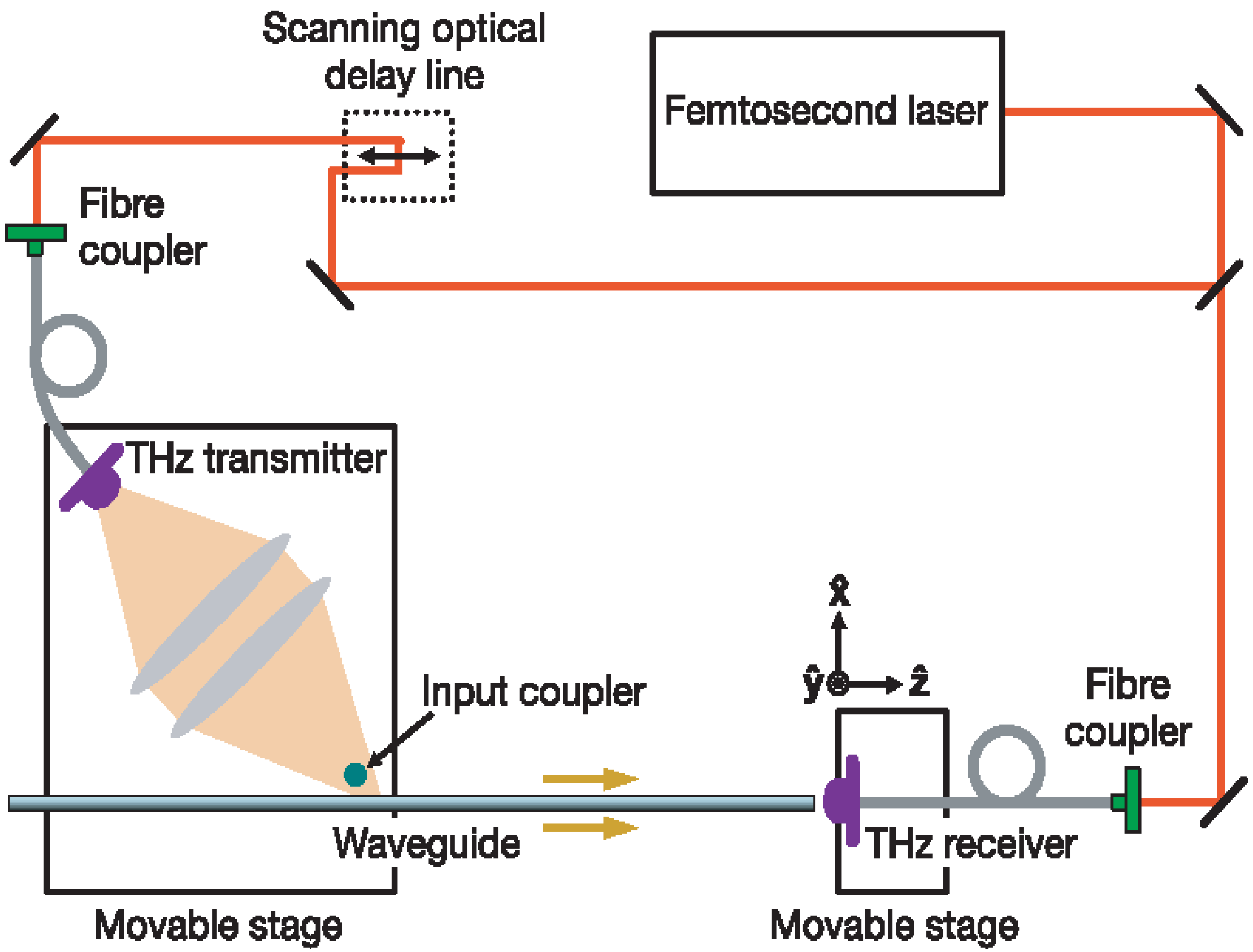
$$\alpha = -\frac{1}{P}\frac{dP}{dz}$$

This is the *power* attentuation constant, i.e.,

$$P \propto e^{-\alpha z}$$

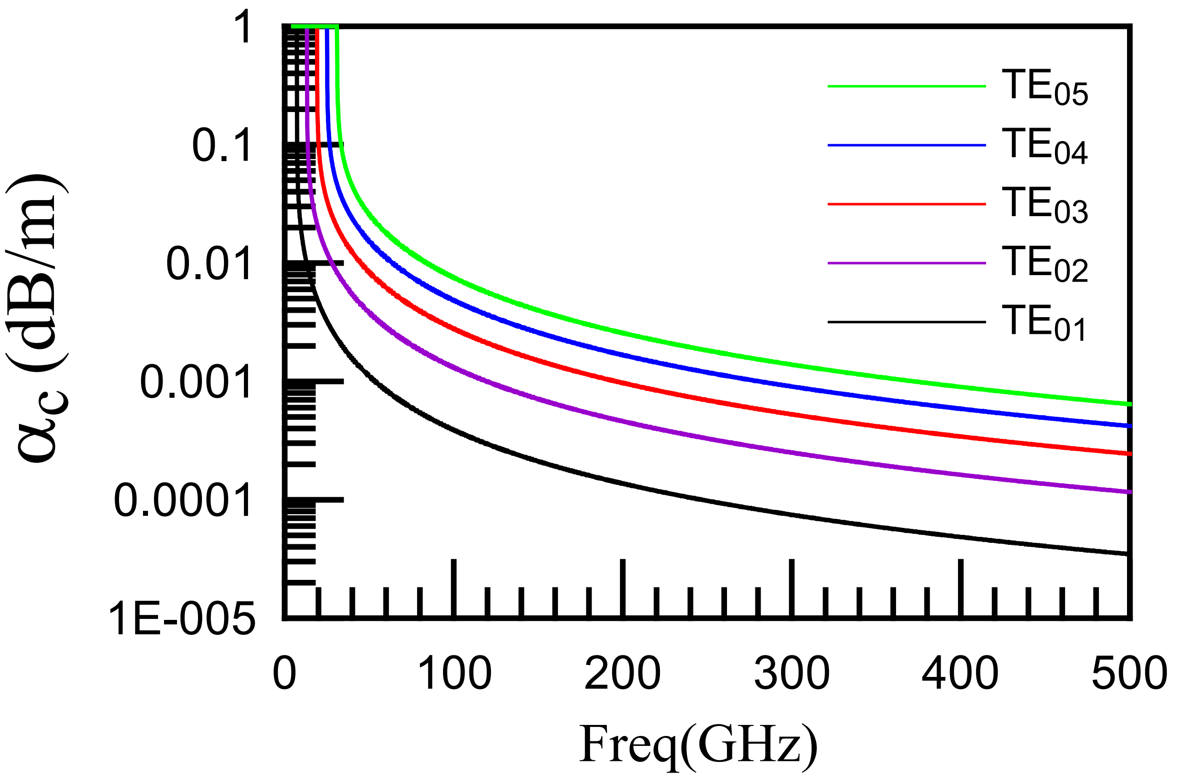
Obviously, the power attentuation constant is twice the value of the field attentuation constant.

Terahertz Waveguide (I)



K. Wang and D. M. Mittleman, "Metal wires for terahertz wave guiding", Nature, vol.**432**, No. 18, p.376, 2004.

Terahertz Waveguide (II):
Using The Lowest Lossy TE01 Mode



References
1. Pozar, p.161.
2. Collin, p.197.

Q: How to excite the TE01 mode and fabricate it at the terahertz region?
A possible solution: X-ray micro-fabrication (LIGA).

H. Y. Yao, J. Y. Jiang, Y. S. Cheng, Z. Y. Chen, T. H. Her, and T. H. Chang*, "Modal analysis and efficient coupling of TE01 mode in small-core THz Bragg fibers", *Optics Express*, 23(21), 27266 (2015).

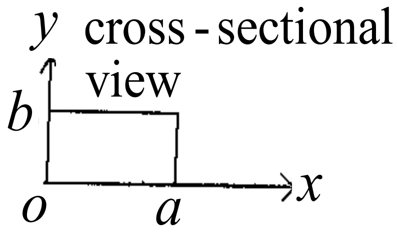
8.7 Modes in Cavities

We consider the example of a rectangular cavity (i.e., a rectangular waveguide with two ends closed by conductors), for which we have two additional boundary conditions at the ends.

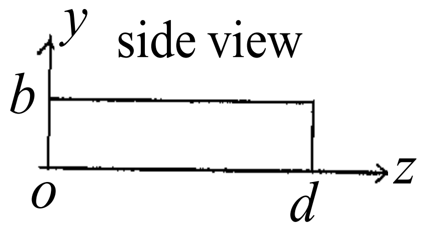
Rewrite (27): $H_z = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\left[C_+ e^{ik_z z - i\omega t} + C_- e^{-ik_z z - i\omega t}\right]$

b.c. (i): $H_z(z=0)=0 \;\Rightarrow\; C_+ = -C_-$

$$\Rightarrow H_z = H_{z0}e^{-i\omega t}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin k_z z$$



b.c. (ii): $H_z(z=d)=0$



$$\Rightarrow \sin k_z d = 0 \;\Rightarrow\; k_z = \frac{l\pi}{d},\; l = 1, 2, \ldots \qquad (32)$$

$$\Rightarrow H_z = H_{z0}e^{-i\omega t}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\sin\frac{l\pi z}{d}, \quad \begin{bmatrix} m, n = 0, 1, 2, \ldots \\ l = 1, 2, \ldots \end{bmatrix} \qquad (33)$$

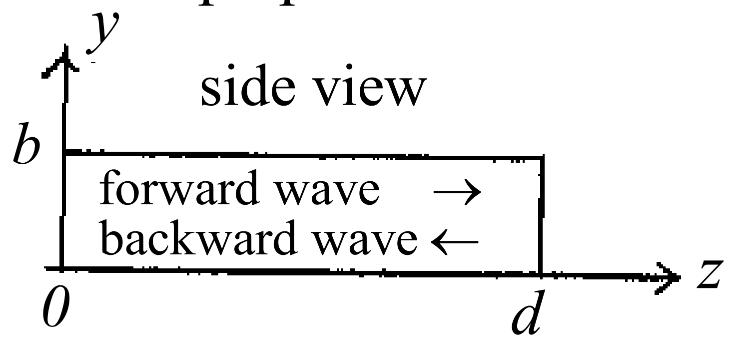
Substituting (32) into $\omega^2 - k_z^2 c^2 - \omega_{cmn}^2 = 0$, where $\omega_{cmn} = \pi c(\frac{m^2}{a^2} + \frac{n^2}{b^2})^{\frac{1}{2}}$

$$\Rightarrow \omega = \omega_{mnl} = \pi c(\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{l^2}{d^2})^{1/2} \quad \left[\begin{array}{l}\omega_{mnl}: \text{resonant frequency} \\ \quad\;\; \text{of the TE}_{mnl} \text{ mode}\end{array}\right] \qquad (34)$$

$$C_+ = -C_-$$

$$\text{From (26)}: \; H_z = \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\left[C_+ e^{ik_z z - i\omega t} + C_- e^{-ik_z z - i\omega t}\right],$$

we see that a cavity mode is formed of a forward wave and a backward wave of equal amplitude. The forward wave is reflected at the right end to become a backward wave, and turns into a forward wave again at the the left end. The forward and backward waves superpose into a standing wave [see (33)]. Thus, we may obtain the other components of the cavity field by superposing the other components of the two traveling waves, as in (26).



Comparison with vibrational modes of a string:

	dependent variable(s)	independent variables	mode index
string	x (oscillation amp.)	$z,\ t$	l
cavity	$E_x,\ E_y,\ B_x,\ B_y,$ E_z (or B_z)	$x,\ y,\ z,\ t$	$m,\ n,\ l$

8.8 Cavity Power Loss and Q

Definition of Q: We have so far assumed a real ω for EM waves in infinite space or a waveguide. Since fields are *stored* in a cavity, it damps in time if there are losses, represented by a complex ω. Thus, fields at any point in the cavity have the time dependence given by

$$E(t) = \begin{cases} E_0 e^{-i\omega_0 t}, & \sigma = \infty \\ E_0 e^{-i(\omega_0 + \Delta\omega)t - \frac{\omega_0}{2Q}t}, & \sigma \neq \infty \end{cases} \quad (8.88)$$

where ω_0 is the resonant frequency [e.g. (34)] without the wall loss.

(8.88) assumes that the wall loss modifies ω_0 by a small real part $\Delta\omega$ and a small imaginary part $\frac{\omega_0}{2Q}$, where $\Delta\omega$ and Q are to be derermined.

Physical reason for $\Delta\omega$: Effective cavity size increases by an amount ~ skin depth δ. A larger cavity has a lower frequency. Hence, $\Delta\omega < 0$.

Physical reason for Q: power dissipation on the wall

$U=$ stored energy in the cavity $[\propto |E|^2 \propto e^{-i\omega t}\cdot e^{i\omega^* t} = e^{2\omega_i t} = e^{-\frac{\omega_0 t}{Q}}]$

$= U_0 e^{-\frac{\omega_0 t}{Q}}$

$\Rightarrow \dfrac{dU}{dt} = -\dfrac{\omega_0}{Q} U$ (power loss)

$$\mathbf{E}(t) = E_0 e^{-i(\omega_0+\Delta\omega)t - \frac{\omega_0}{2Q}t}$$
$$\Rightarrow \omega_i = -\frac{\omega_0}{2Q} \quad (8.87)$$

$\Rightarrow Q = \omega_0 \dfrac{\text{stored energy}}{\text{power loss}}$ (time-domain definition of Q) (8.86)

(8.88) represents a damped oscillation which does not have a single frequency. To exame the frequency of $E(t)$, we write

$$E(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} E(\omega) e^{-i\omega t} d\omega,$$

where

Use (8.88), assume $E(t) = 0$ for $t < 0$

$$E(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} E(t) e^{i\omega t} dt = \frac{1}{\sqrt{2\pi}} E_0 \int_0^{\infty} e^{-\frac{\omega_0}{2Q}t + i(\omega-\omega_0-\Delta\omega)t} dt$$

$$= \frac{1}{\sqrt{2\pi}} \frac{E_0}{-i(\omega-\omega_0-\Delta\omega)+\frac{\omega_0}{2Q}}$$

The frequency spectrum is best seen from the field energy distribution in frequency-domain

$$|E(\omega)|^2 \propto \frac{1}{(\omega-\omega_0-\Delta\omega)^2+\left(\frac{\omega_0}{2Q}\right)^2} = \begin{cases} \max, & \omega=\omega_0+\Delta\omega \\ \frac{1}{2}\max, & \omega=\omega_0+\Delta\omega\pm\frac{\omega_0}{2Q} \end{cases} \quad (8.90)$$

$$\Rightarrow \quad \delta\omega = \begin{bmatrix} \text{full width at} \\ \text{half-maximum points} \end{bmatrix} = \frac{\omega_0}{Q}$$

$$\Rightarrow \quad Q = \frac{\omega_0}{\delta\omega} \quad \text{(frequency-domain definition of } Q\text{)} \quad (8.91)$$

Note: ω_0 is the resonant frequency of the cavity in the absence of any loss. $\omega_0+\Delta\omega$ is the resonant frequency in the presence of losses. In most cases, the difference is insignificant.

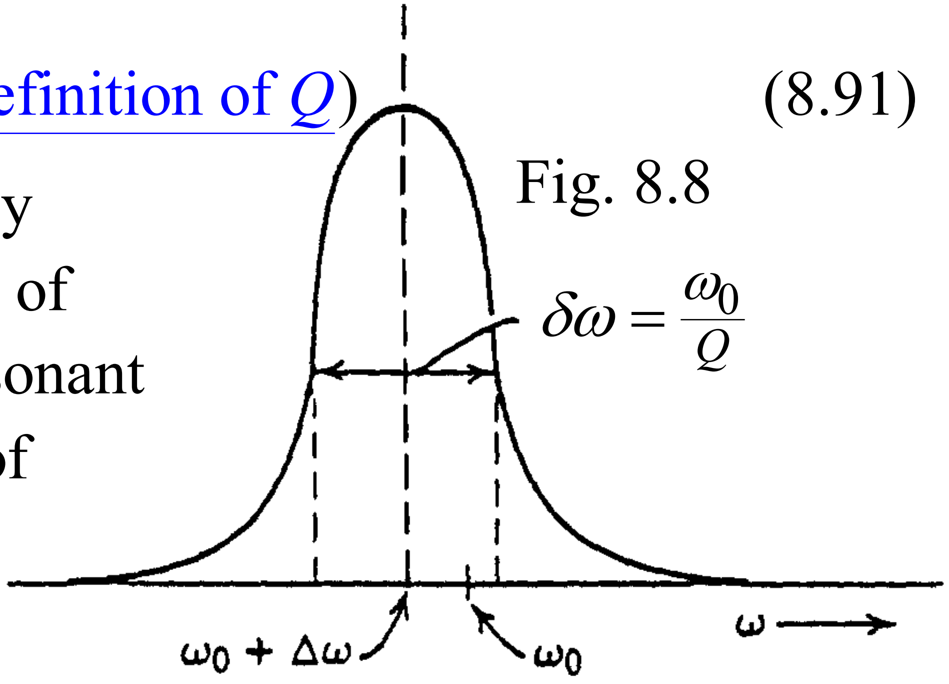


Fig. 8.8

Physical Interpretation of Q :

(i) Use the time-domain definition: $Q = \omega_0 \dfrac{\text{stored energy}}{\text{power loss}}$

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{\tau_0}$$

τ_0: wave period

$$\frac{\text{stored energy}}{\text{power loss}} \approx \tau_d$$

τ_d: decay time of stored energy

$$\Rightarrow Q = \omega_0 \frac{\text{stored energy}}{\text{power loss}} \approx 2\pi \frac{\tau_d}{\tau_0} \tag{48}$$

(48) shows that Q, which results from the power loss, is approximately 2π times the number of oscillations during the decay time. A larger Q value implies that the field energy can be stored in the cavity for a longer time. Hence, Q is often referred to as the quality factor.

(ii) Use the frequency-domain definition: $Q = \frac{\omega_0}{\delta\omega}$ (see Fig. 8.8)

For a lossy cavity, a resonant mode can be excited not just at one frequency (as is the case with a lossless cavity) but at a range of frequencies ($\delta\omega$). The resonant frequency ($\omega_0+\Delta\omega$, see Fig. 8.8) of a lossy cavity is the frequency at which the cavity can be excited with the largest inside-field amplitude, given the same source power. The resonant width $\delta\omega$ of a mode is equal to the resonant frequency divided by the Q value of that mode (see Fig. 8.8). Note that each mode has a different Q value.

Figure 8.8 can be easily generated in experiment to measure the Q value.

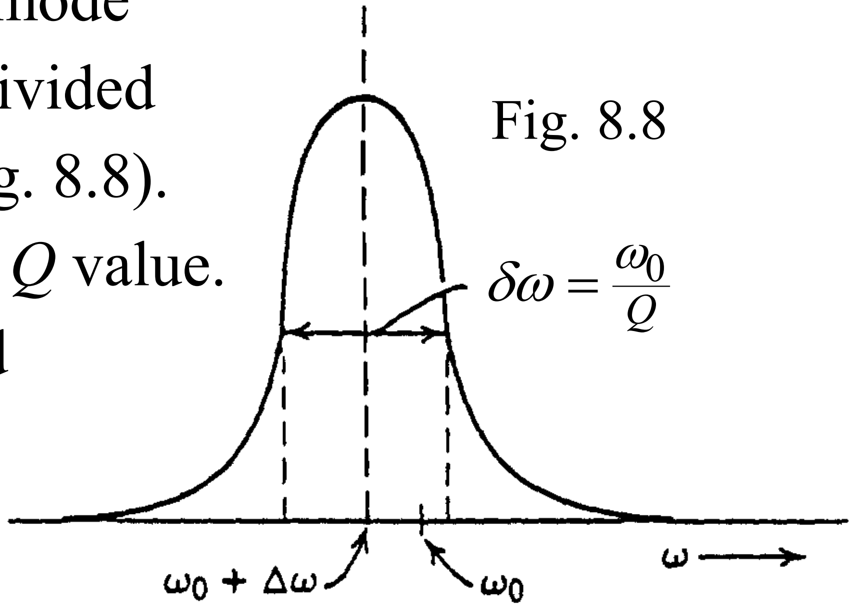


Fig. 8.8

$$Q = \omega_0 \frac{\text{stored energy}}{\text{power loss}}$$

Using the results of Sec. 8.1, we can calculate Q (but not $\Delta\omega$) due to the Ohmic loss. We first calculate the zero order **E** and **H** of a specific cavity assuming $\sigma = \infty$, then use the zero order **E** and **H** to calculate U and power loss,

$$\text{stored energy} = \int_V (w_e + w_m) d^3x = \begin{cases} 2\int_V w_e d^3x = \frac{\varepsilon}{2}\int_V |\mathbf{E}|^2 d^3x \\ 2\int_V w_m d^3x = \frac{\mu}{2}\int_V |\mathbf{H}|^2 d^3x \end{cases}$$

(6.133)

(8.15)

$$\text{power loss} = \frac{1}{2\sigma\delta}\oint_S |\mathbf{K}_{eff}|^2 da$$

$$= \frac{1}{2\sigma\delta}\oint_S |\mathbf{n}\times\mathbf{H}|^2 da$$

(8.14)

Formulae for Q (due to ohmic loss) for rectangular and cylindrical cavities can be found in, for example, R. E. Collin, "Foundation of Microwave Engineering", p. 503 and p. 506.

***Q* due to other types of losses :** If there are several types of power losses in a cavity (e.g., due to Imε and coupling losses), Q can be expressed as follows:

$$Q = \omega_0 \frac{\text{stored energy}}{\sum_n (\text{power loss})_n} \qquad (49)$$

n-th type of power loss

$$\Rightarrow \frac{1}{Q} = \sum_n \frac{1}{Q_n} \qquad (50)$$

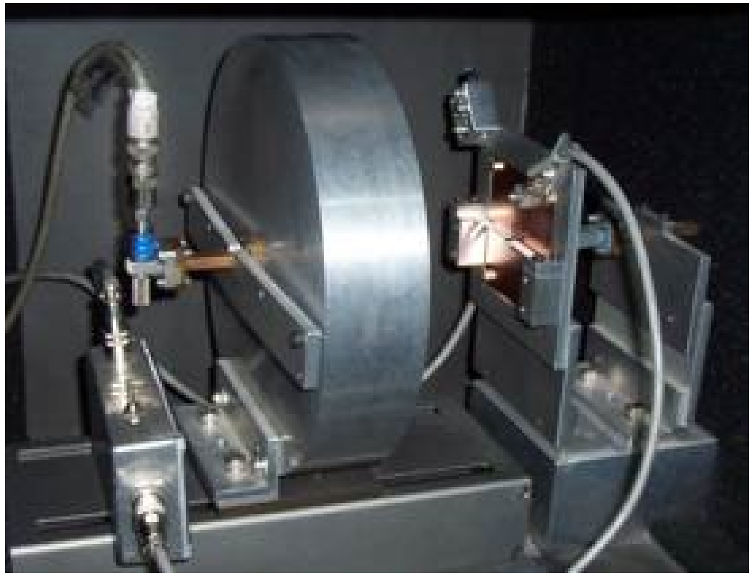
where Q_n (Q due to the n-th type of power loss) is given by

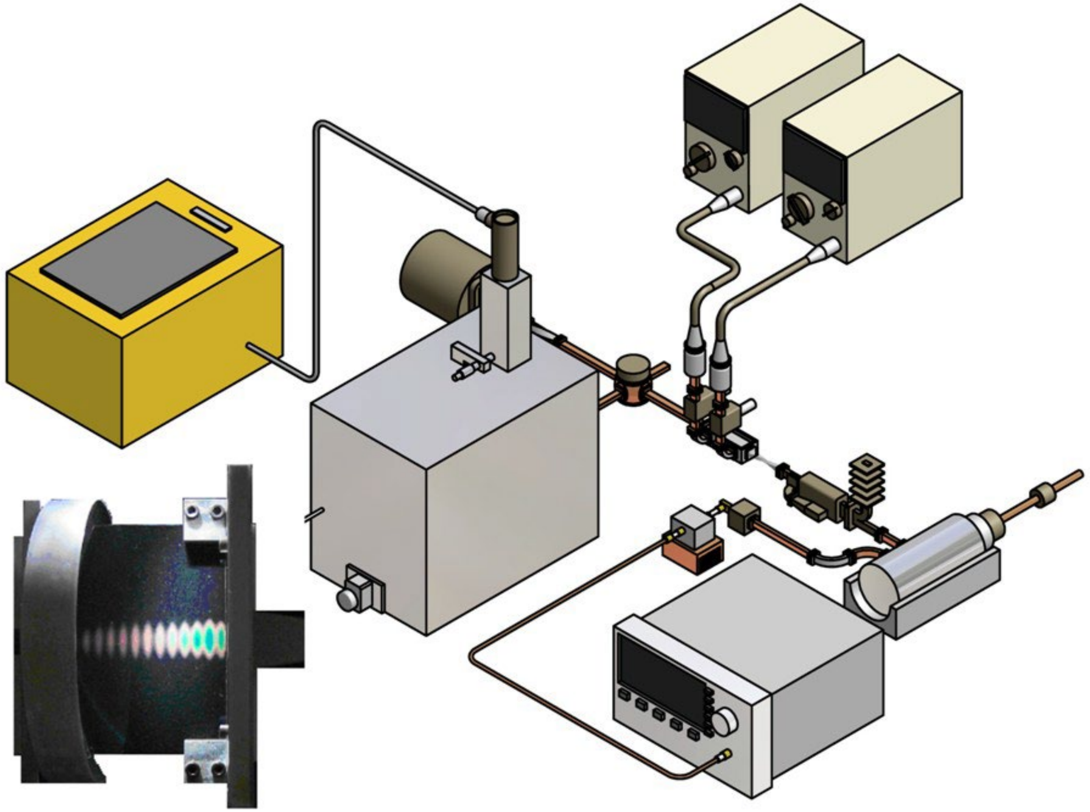
$$Q_n = \omega_0 \frac{\text{stored energy}}{(\text{power loss})_n}$$

A Comparison between Waveguides and Cavities

	Waveguide	Cavity
Function	transport EM energy	store EM energy
Characteri-zation	dispersion relation and attenuation constant	resonant frequency and Q
Examples of applications (mostly for microwaves, 0.3-300 GHz)	transport of high power microwaves (such as multi-kW waves for long-range radars and communi-cations)	(1) particle acceleration (2) frequency measurement (3) material processing

High-Q Microwave/Material Applicator





$$Q = \omega_0 \frac{\text{stored energy}}{\sum_n (\text{power loss})_n}$$

Conductor loss, dielectric loss, radiation loss, diffraction loss…

Homework of Chap. 8 Problems: 2, 3, 4

Problem 8.2

A transmission line consisting of two concentric circular cylinders of metal with conductivity σ and skin depth δ, as shown, is filled with a uniform lossless dielectric (μ,ε). A TEM mode is propagated along this line. Section 8.1 applies.

(a) Show that the time averaged power flow along the line is

$$P=\sqrt{\frac{\mu}{\varepsilon}}\pi a^2 |H_0|^2 \ln\left(\frac{b}{a}\right)$$

where H_0 is the peak value of the azimuthal magnetic field at the surface of the inner conductor.

(b) Show that the transmitted power is attenuated along the line as

$$P(z)=P_0 e^{-2\gamma z}$$

where

$$\gamma=\frac{1}{2\sigma\delta}\sqrt{\frac{\mu}{\varepsilon}}\frac{\left(\frac{1}{a}+\frac{1}{b}\right)}{\ln\left(\frac{a}{b}\right)}$$

(c) The characteristic impedance Z_0 of the line is defined as the ratio of the voltage between the cylinders to the axial current flowing in one of them at any position Z. Show that for this line

$$Z_0=\frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln\left(\frac{b}{a}\right)$$

(d) Show that the series resistance and inductance per unit length of the line are

$$R=\frac{1}{2\pi\sigma\delta}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$L=\left\{\frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)+\frac{\mu_c\delta}{4\pi}\left(\frac{1}{a}+\frac{1}{b}\right)\right\}$$

where μ_c is the permeability of the conductor. The correction to the inductance comes from the penetration of the flux into the conductors by a distance of order δ.

Problem 8.3

(a) A transmission lineconsists of two identical thin strips of metal, shown in cross section in thesketch. Assuming that $b >> a$, discuss the propagation of a TEM mode on this line, repeating the derivationsof Problem 8.2. Show that

$$P=\frac{ab}{2}\sqrt{\frac{\mu}{\varepsilon}}|H_0|^2$$

$$\gamma=\frac{1}{a\sigma\delta}\sqrt{\frac{\varepsilon}{\mu}}$$

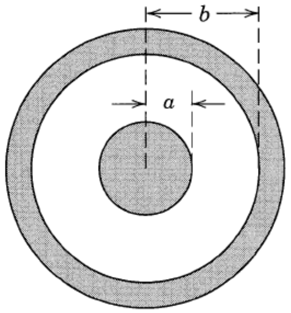
$$Z_0=\sqrt{\frac{\mu}{\varepsilon}}\left(\frac{a}{b}\right)$$

$$R=\frac{2}{a\delta b}$$

$$L=\left(\frac{\mu a+\mu_c\delta}{b}\right)$$

where the symbols on the lefthave the same meanings as in Problem 8.2.

(b) The lower half of thefigure shows the cross section of a microstrip line with a strip of width b mounted on a dielectric substrate ofthickness h and dielectric constant ε, all on a ground plane.
What differences occur here compared to part a if $b >> h$? If $b << h$?



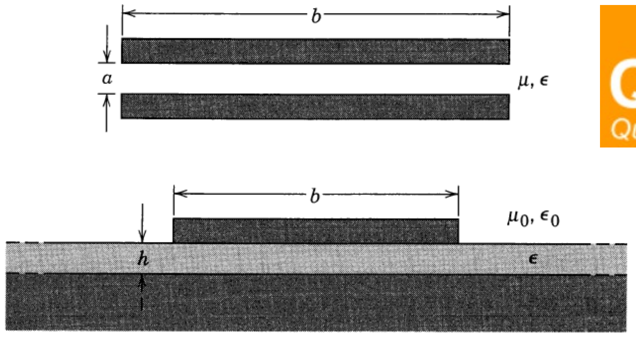
Problem 8.2

Problem 8.4

Transverse electric and magnetic waves arepropagated along a hollow, right circular cylinder with inner radius R and conductivity σ.

(a) Find the cutoff frequencies of the various TE and TM modes. Determine numerically the lowest cutoff frequency (the dominant mode) in terms of the tube radius and the ratio of cutoff frequencies of the next four higher modes to that of the dominant mode. For this pat assume that the conductivity of the cylinder is infinite.

(b) Calculate the attenuation constants of the waveguide as a function of frequency for the lowest two distinct modes and plot them as a function of frequency.



Problem 8.3

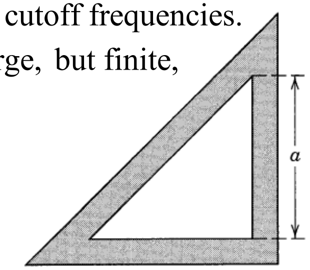
Homework of Chap. 8 Problems: 5, 6, 18

Problem 8.5

A waveguide is constructed so that the cross section of the guide forms a right triangle with sides of length a, a, $\sqrt{2a}$, as shown. The medium inside has $\mu_r = \varepsilon_r = 1$.

(a) Assuming infinite conductivity for the walls, determine the possible modes of propagation and their cutoff frequencies.

(b) For the lowest modes ofeach type calculate the attenuation constant, assuming that the walls havelarge, but finite, conductivity. Compare the result with that for a squareguide of side a made from the same material.



Problem 8.5

Problem 8.6

A resonant cavity of copper consists of a hollow, right circular cylinder of inner radius R and length L, with flat end faces.

(a) Determine the resonant frequencies of the cavity for all types of waves. With $(1/\sqrt{\mu\varepsilon}R)$ as a unit of frequency, plot the lowest four resonant frequencies of each type as a function of R/L of $0 < R/L < 2$.

Does the same mode have the lowest frequency for all R/L?

(b) If $R = 2$ cm, $L = 3$ cm, and the cavity is made of pure copper, what is the numerical value of Q for the lowest resonant mode?

Problem 8.18

(a) From the use of Green'stheorem in two dimensions show that the TM and TE modes in a waveguide defined by the boundary – value problems (8.34) and (8.36) are orthogonal in the sense that

$$\int_A E_{z,\lambda} E_{z,\mu} da = 0 \text{ for } \lambda \neq \mu$$

for TM modes, and a correspondingrelation for H_z for TE modes.?

(b) Prove that the relations (8.131) – (8.134) form a consistent set of normalization conditions for thefields, including the circumstances when λ is a TM mode and μ is a TE mode.

Optional Homework of Chap. 8 Problems: 19, 20

Problem 8.19

The figure shows a cross – sectional view of an infinitely long rectangular waveguide with the center conductor of a coaxial the line extending vertically a distance h into its interior at $z = 0$. The current along the probe oscillates sinusoidally in time with frequency ω, and its variation in space can be approximated as $I(y) = I_0 \sin\left[\left(\frac{\omega}{c}\right)(h-y)\right]$. The thickness of the probe can be neglected. The frequency is such that only the TE_{10} mode can propagate in the guide.

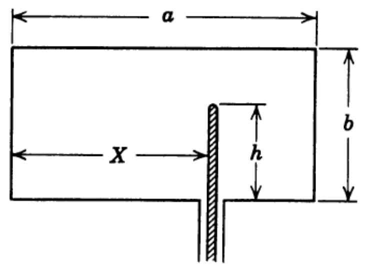
(a) Calculate the amplitudes for excitation of both TE and TM modes for all (m,n) and show how the amplitudes depend on m and n for $m,n >> 1$ for a fixed frequency ω.

(b) For the propagating mode show that the power radiated in the positive z direction is

$$P = \frac{\mu c^2 I_0^2}{\omega kab} \sin^2\left(\frac{\pi X}{a}\right) \sin^4\left(\frac{\omega h}{2c}\right)$$

with an equal amount in the opposite direction. Here k is the wave number for the TE_{10} mode.

(c) Discuss the modifications that occur if the guide, instead of running off to infinity in both directions, is terminated with a perfectly conducting surface at $z = L$. What values of L will maximize the power flow for a fixed current I_0? What is the radiation resistance of the probe (defined as the ratio of power flow to one-half the square of the current at the base of the probe) at maximum?



Problem 8.19

Problem 8.20

An infinitely long rectangular waveguide has a coaxial line terminating in the short side of the guide with the thin central conductor forming a semicircular loop of radius R whose center is a height h above the floor of the guide, as shown in the accompanying cross-sectional view. The half-loop is in the plane z = 0 and its radius R is sufficiently small that the current can be taken as having a constant value I_0 everywhere on the loop.

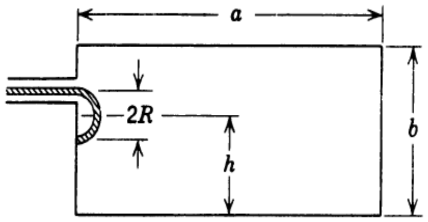
(a) Prove that to the extent that the current is constant around the half-loop, the TM modes are not excited. Give a physical explanation of this lack of excitation.

(b) Determine the amplitudefor the lowest TE mode in the guide and show that its value is independent ofthe height h.

(c) Show that the powerradiated in either direction in the lowest TE mode is

$$P = \frac{I_0^2}{16} Z \frac{a}{b} \left(\frac{\pi R}{a} H\right)^4$$

where Z is the wave impedanceof the TE_{10} mode. Here assume $R << a, b$.



Problem 8.20